New Gravitational Memories

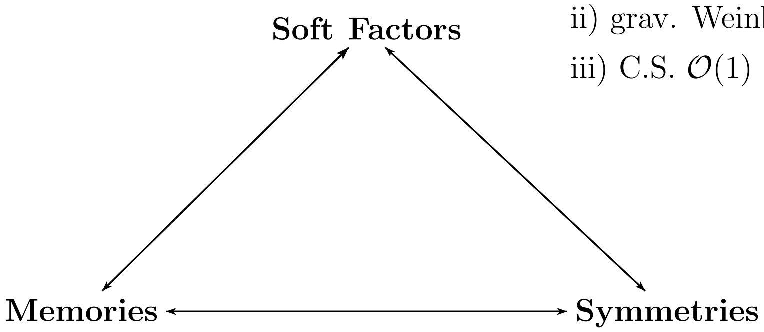
I Agenda

- SMS Outline
- Interlude A) Asymptotic observers
- Round 1: E&M
- Interlude B) Asymptotic symmetries
- Round 2: Standard Memory
- Round 3: Spin Memory

The goal of this talk is to describe the intuition that led to the 'spin memory effect.' It will serve as a primer to the upcoming P.S.Z. paper, as well as (hopefully) give some appreciation for what Andy and friends have been doing this past year.

II SMS Outline

i) E&M Weinberg $\mathcal{O}(\frac{1}{\omega})$
ii) grav. Weinberg $\mathcal{O}(\frac{1}{\omega})$
iii) C.S. $\mathcal{O}(1)$



i) Liénard-Wiechert
ii) C.K.
iii) P.S.Z.

Global	Asymptotic
i) e-charge	large gauge transformations
ii) p^μ	supertranslations
iii) $J^{\mu\nu}$	superrotations

III Interlude A) Asymptotic observers

- conventions
- connection between position space (large r) and momentum space (small ω)

The conformal compactification of Minkowski space gives a nice way to visualize the causal structure. The diagram is drawn so that the left side vs. right side are the NP and SP of a full S^2. The retarded and advanced times

$$u = t - r, \quad v = t + r \tag{III.1}$$

give the metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 = -dudv + \frac{(v-u)^2}{4} d\Omega^2. \tag{III.2}$$

Now letting

$$u = L \tan U, \quad v = L \tan V \tag{III.3}$$

gives

$$ds^2 = \frac{L^2}{\cos^2 U \cos^2 V}(-dUdV + \frac{\sin^2(U-V)}{4} d\Omega^2) \tag{III.4}$$

where $U, V \in \{-\frac{\pi}{2}, \frac{\pi}{2}\}$.

Massless particles come in along past null infinity $\mathcal{I}^-$ and exit at future null infinity $\mathcal{I}^+$. In this compactification, spatial infinity i^0 is a single point. This is one way of motivating why

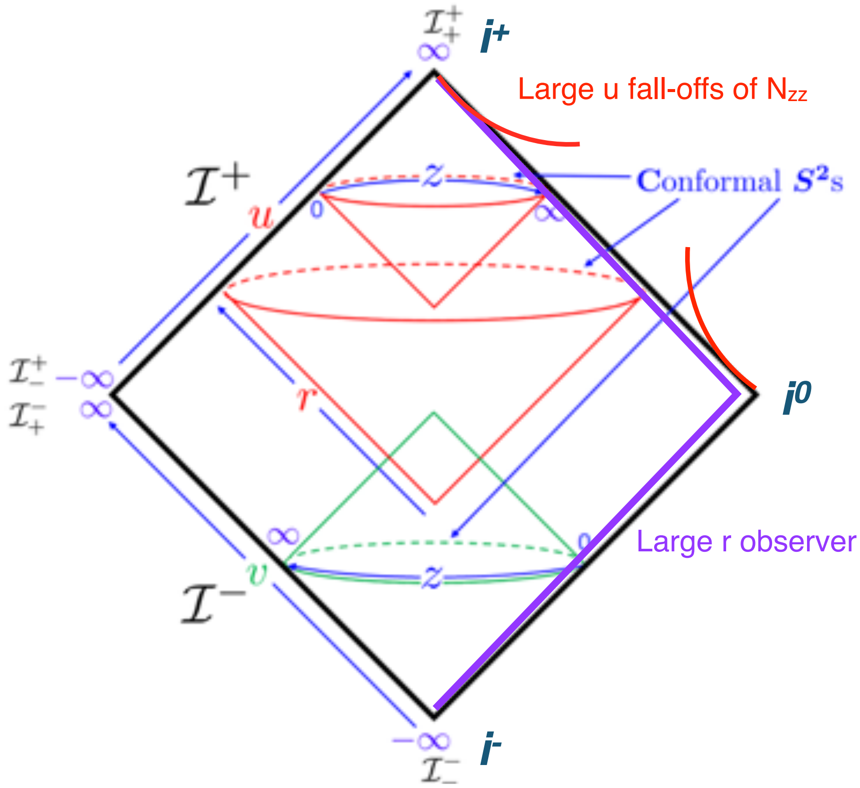
III.I Conventions

I will parameterize coordinates on the sphere with

$$z = e^{i\phi} \tan\frac{\theta}{2}, \quad d\Omega^2 = 2\gamma_{z\bar{z}} dz d\bar{z}, \quad \gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2} \tag{III.5}$$

I will be considering on-shell particles, whose momenta are described by their energy and a point on the sphere

$$\begin{aligned} p^\mu &= E(1, \hat{x}) && \text{massless} \\ p^\mu &= m\gamma(1, \beta\hat{x}) && \text{massive} \end{aligned} \tag{III.6}$$



When one looks at the mode expansion for a massless field, for example the gauge field

$$\mathcal{A}_\mu(x) = e \sum_{\alpha=\pm} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_q} [\varepsilon_\mu^{\alpha *} a_\alpha(\vec{q}) e^{iq\cdot x} + h.c.] \tag{III.7}$$

one notices from the phase

$$e^{iq\cdot x} = e^{-i\omega u - i\omega r(1-\hat{x}\cdot\hat{q})} \tag{III.8}$$

that taking the large r limit will pick out $\hat{q} = \hat{x}$. This stationary phase approximation relates the low frequency momentum space behavior to the large distance position space behavior at the same point on a sphere.

This position-to-momentum space interpretation becomes manifest in soft factors. When you have a set of *in* and *out* particles in on-shell momentum eigenstates, the amplitude for an n particle scattering configuration is related to an $n+1$ soft gauge field

$$\langle out|a_\alpha(q)\mathcal{S}|in\rangle = (S^{(0)\alpha} + S^{(1)\alpha} + ...)\langle out|\mathcal{S}|in\rangle \tag{III.9}$$

where the first term is $\mathcal{O}(\frac{1}{\omega})$ and the subleading terms go down in powers of ω. Normally these soft factors are derived by taking Feynman diagrams and allowing the additional low momentum photon to attach to all external legs. Vertex factors from the interactions determine the numerator, while the additional propagator gives the pole for the leading term.

These can equivalently be thought of as classical equations of motion, connecting soft factors to memories.

IV Round 1: E&M

- radiation zone
- soft factor and expectation values
- intuition for boundary matching

A natural question to ask is how the $\mathcal{I}^+$ description carries over to large-but-finite r observers. While on the Penrose diagram such an observer looks like it skirts $\mathcal{I}^+$, it is on a timelike, rather than a null, trajectory. The answer is in the order of limits. By imposing fall off conditions on the dynamical variables, one can focus on dynamics where $u << r$.

The simplest example is to consider the radiation sourced by an accelerated charge in electromagnetism. The equation governing such radiation

$$\partial_u A_u = \partial_u (D^{\bar{z}} A_{\bar{z}} + D^z A_z) + e^2 j_u \tag{IV.1}$$

when integrated over time is a statement about the final versus initial charge-velocity distributions. With the identification

$$\Delta A_z = -\frac{1}{4\pi} \hat{\varepsilon}_z^{*+} \omega S^{(0)+} \tag{IV.2}$$

coming from the mode expansion, saying that the soft factor gives the expectation value of ΔA_z means that the radial component of the electric field has the following contribution from a single boosted charge:

$$\lim_{\omega \to 0} \omega [D^z \hat{\varepsilon}_z^{*+} S_p^{(0)+} + D^{\bar{z}} \hat{\varepsilon}_{\bar{z}}^{*-} S_p^{(0)-}] \quad = \tfrac{Q}{\gamma^2} \tfrac{1}{(1-\vec{\beta}\cdot\hat{n})^2} \tag{IV.3}$$

where

$$S_p^{(0)\pm} = Q \frac{p \cdot \varepsilon^{\pm}}{p \cdot q} \tag{IV.4}$$

Note that simply plugging in the soft factor gives the electric field for a perpetually moving particle. In terms of the position of the source at the retarded time, the Liénard-Weichert potential gives

$$E_r = \frac{q}{4\pi}\frac{1}{\gamma^2(1-\vec{\beta}\cdot\hat{n})^2|r-r_s|^2} \tag{IV.5}$$

which is in exact agreement with plugging massive p^μ into the soft factor. This also yields two important insights. 1. We can equivalently write an expression for the electric field of a perpetually moving charge as a function of its current position. In this case

$$\vec{E} = \frac{q}{4\pi}\frac{1}{\gamma^2(1-\beta^2\sin^2\theta)^{\frac{3}{2}}}\frac{\hat{R}}{R^2} \tag{IV.6}$$

If the charge moves through the spacetime origin, towards the north pole for instance, then a little bit after $t=0$, it will be closer to the NP, whereas a little bit before, it will be closer to the SP. The radial electric field thus obeys a natural "antipodal" matching of sorts. The issue of matching across i^0 is non-trivial, but one would hope that in the finite r limit, one can still make sense of this type of matching, so an argument like this is helpful. 2. We can think of these zero-mode position space measurements as telling us something about what dynamics had to happen (i.e. radiation) given what came out versus what went in, when considered as two separate static configurations.

Massless scattering a la LSET. Fields localized to shell emerging from spacetime origin along $u=0$. Start in vacuum, massless charges pass, vector potential changes as a result. Between step functions, looks like "pure gauge" on the sphere. Large gauge transformations effectively allow you to "reset" your gauge field between any two wave fronts.

V Interlude B) Asymptotic symmetries

- Bondi Gauge
- supertranslation and superrotation vector fields
- G_{uu} and G_{uz} constraint equations

Asymptotically flat metrics in Bondi Gauge have the form

$$\begin{aligned} ds^2 &= -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} + 2\tfrac{m_B}{r}du^2 \\ &\quad + \left(rC_{zz}dz^2 + D^zC_{zz}dudz + \tfrac{1}{r}\left(\tfrac{4}{3}N_z - \tfrac{1}{4}\partial_z(C_{zz}C^{zz})\right)dudz + c.c.\right) + ... \end{aligned} \tag{V.1}$$

Key things to note about this gauge is that $g_{rr} = g_{ra} = 0$. C_{zz} is a complex degree of freedom. The G_{uu} and G_{ua} $\mathcal{O}(r^{-2})$ constraint equations

$$G_{\mu\nu} = 8\pi G T^M_{\mu\nu} \tag{V.2}$$

relate these to the Bondi mass and angular momentum aspect. From G_{uu}

$$\begin{aligned} \partial_u m_B &= \tfrac{1}{4}\left[D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}\right] - T_{uu}, \\ T_{uu} &\equiv \tfrac{1}{4} N_{zz} N^{zz} + 4\pi G \lim_{r\to\infty}[r^2 T^M_{uu}], \end{aligned} \tag{V.3}$$

Taking $\partial_{\bar{z}}$ of the G_{uz} constraint and ∂_z of the complex conjugate $G_{u\bar{z}}$ constraint gives:

$$\partial_z \partial_{\bar{z}} m_B = \mathrm{Re}[\partial_u \partial_{\bar{z}} N_z + \partial_{\bar{z}} T_{uz}] \tag{V.4}$$

$$\mathrm{Im}[\partial_{\bar{z}} D_z^3 C^{zz}] = 2\mathrm{Im}[\partial_u \partial_{\bar{z}} N_z + \partial_{\bar{z}} T_{uz}]. \tag{V.5}$$

There are two families of vector fields which preserve the asymptotic form of the metric: supertranslations parameterized by an arbitrary scalar on the sphere and superrotations corresponding to a Virasoro symmetry parametrized by a conformal killing vector.

$$\begin{aligned} \xi^+ =& (1+\frac{u}{2r})Y^{+z}\partial_z - \frac{u}{2r}D^{\bar{z}}D_z Y^{+z}\partial_{\bar{z}} - \frac{1}{2}(u+r)D_z Y^{+z}\partial_r + \frac{u}{2}D_z Y^{+z}\partial_u + c.c. \\ &+ f^+\partial_u - \frac{1}{r}(D^z f^+\partial_z + D^{\bar{z}} f^+\partial_{\bar{z}}) + D^z D_z f^+\partial_r, \end{aligned} \tag{V.6}$$

(Note that in the limit that one is near the $u=0$ lightcone, the lorentz transformations look like global conformal transformations of the sphere).

There are two limits in which solving the radiation constraint equations simplifies. 1) Assume only outgoing radiation sourced by changes in the distribution of massive matter. Then at linear order, the change on the left hand side gives the metric perturbations on the right. 2) Assume only massless in and out states, then use boundary conditions to fix the early and late time limits of the left hand side. This sets the stage for the scattering problem and $\mathcal{S}$-matrix connection.

These symmetries are turned into ward identities for massless scattering processes as follows:

- set boundary conditions and matching between $\mathcal{I}^-$ and $\mathcal{I}^+$.
- the Lie derivative gives the action of the symmetry on the hard particles
- limits to zero mode radiative brackets (special to d=4) motivate the conjugate zero-

mode operator

- the lie derivative acting on the flat metric gives the inhomogeneous change generated in the linear theory
  - $\delta_Y C_{zz} = -uD_z^3 Y^z + ... \rightarrow Q_S^+ = -\frac{1}{2}\int dud^2zD_z^3Y^z u\partial_u C^z_{\bar{z}}$
  - $\delta_f C_{zz} = -2D_z^2 f + ... \rightarrow$ soft graviton current involving $\int du\partial_u C_{zz}$

verifying that the charges generated the symmetry via the soft factor amounted to showing that the soft operator gave the appropriate Lie derivative acting on the hard states. (For the super translation case, the charge reduced to an expression like the Bondi Mass aspect in the first constraint equation.)

One important boundary condition worth mentioning here is that the curl of C_{zz} vanishes at the limits of $\mathcal{I}^+$. This makes it possible for its integral to be finite, which we will see relates to the spin memory effect.

VI Round 2: Standard Memory

- soft factor and boosted mass
- Supertranslation charge $\delta(\omega)$ and $\frac{1}{\omega}$ zero modes
- intepreting the metric

The boundary conditions on the metric imply that at early and late u

$$C_{zz} = -2D_z^2 C \tag{VI.1}$$

much like in the electromagnetic case, this looks like a pure supertranslation, though it really indicates a transition between two different super translated vacuua.

Start in vacuum, shockwave passes, metric changes as a result but is still in an asymptotically flat form. Between wavefronts, looks like nothing is happening. BMS supertranslations effectively allow you to “reset” your metric between any two shock waves.

Here the conjugate zero modes are a shift and a step, with the step describing the Weinberg pole and its conjugacy to the pure supertranslation shift defining the $\mathcal{S}$ matrix ward identity. Comparing the metric at early and late times, one sees that a change in C_{zz} amounts to a change in physical distances between objects at different points on a large r sphere.

S.Z. show that the leading soft factor amounts to the Braginsky Thorne result with the expectation value position-to-momentum-space interpretation. Like the E&M case, on can read off from the constraint equation Δm_B cooresponding to a set of boosted masses, which individually have the form

$$m_B = \frac{m}{\gamma^3} \frac{1}{(1 - \vec{\beta} \cdot \hat{n})^3} \tag{VI.2}$$

The massive soft factor contribution from a single particle

$$S_p^{(0)\pm} = \frac{(p \cdot \varepsilon^{\pm})^2}{p \cdot q} \tag{VI.3}$$

obeys

$$\lim_{\omega \to 0} \omega [D^z D^z \hat{\varepsilon}^{*+}_{zz} S_p^{(0)+} + D^{\bar{z}} D^{\bar{z}} \hat{\varepsilon}^{*-}_{\bar{z}\bar{z}} S_p^{(0)-}] \quad = -\tfrac{m}{\gamma^3} \tfrac{1}{(1 - \vec{\beta} \cdot \hat{n})^3} \tag{VI.4}$$

where $\hat{n}$ is the unit vector pointing parallel to $\vec{q}$ and I have used

$$p^\mu = m\gamma(1, \vec{\beta}). \tag{VI.5}$$

The constraint equation gives:

$$\partial_u m_B = \frac{1}{4} D_z^2 N^{zz} + \frac{1}{4} D_{\bar{z}}^2 N^{\bar{z}\bar{z}} - \frac{1}{2} T^M_{uu} - \frac{1}{4} N_{zz} N^{zz}, \tag{VI.6}$$

so that a linearized theory with no massless matter would have

$$\Delta m_B = \frac{1}{4} [D_z^2 \Delta C^{zz} + D_{\bar{z}}^2 \Delta C^{\bar{z}\bar{z}}]. \tag{VI.7}$$

From the stationary phase approximation, we identify ΔC_{zz} as

$$\Delta C_{zz} = -\frac{\kappa^2}{8\pi} \hat{\varepsilon}^{*+}_{zz} \omega S_p^{(0)+} \tag{VI.8}$$

reproducing the above.

VII Round 3: Spin Memory

- interpreting the metric
- thought experiment: the BMS ring
- rotation of inertial observers and $D^A C_{AB} dx^B$

- Superrotation charge

K.L.P.S. showed that the superrotations corresponded to the subleading soft factor, which essentially came from the soft charge $Q_S^+ = -\frac{1}{2}\int dud^2zD_z^3Y^zu\partial_uC^z{}_{\bar{z}}$ picking out the order 1 soft factor due to the projection $\lim_{\omega\to 0}(1+\omega\partial_\omega)$ coming from the u integral.

This got me thinking about how to naturally construct a u integrated observable. The result was the following thought experiment. Imagine sending light rays around a ring fixed to the BMS observer's coordinates. The path that a beam will travel between two nearby mirrors along this forced trajectory will be described by $ds^2 = 0$. If one looks at the asymptotically flat metric in Bondi gauge, one will note that there are various places where lengths will change, but there is one set of terms, namely the $dudz$ and *c.c.* part that are odd under $\delta z \to -\delta z$. If you have two beams going between the same mirrors in opposite directions at the same time, then this would pick out that correction to the metric.

Now let us consider a small ring. If the metric does not change appreciably over the time scale during which a beam circuits the ring, then over the course of one orbit, each segment is effectively transversed by two beams moving in opposite directions at effectively "same time" as regards the metric fluctuations. (Note that I can always scale the size of the ring down so that this limit holds). Going in one direction is slightly faster than the other and this curl part accumulates so that over the course of one orbit, I get a contour integral and time delay

$$\Delta P = \oint_{\mathcal{C}} \left(D^zC_{zz}dz + D^{\bar{z}}C_{\bar{z}\bar{z}}d\bar{z}\right). \tag{VII.1}$$

Now if I repeat the orbit again, the time delay accumulates. In the limit where the repetition is much faster than the time delay, the net time accumulated delay is simply the time integral of the curl of the metric divided by the period of the beam orbit:

$$\Delta^+ u = \frac{1}{2\pi L}\int du \oint_{\mathcal{C}} \left(D^zC_{zz}dz + D^{\bar{z}}C_{\bar{z}\bar{z}}d\bar{z}\right). \tag{VII.2}$$

This has multiple nice features, the most important of which is that it is insensitive to supertranslations, or even superrotations for that matter. The final state of the metric will be such that this curl is zero, and this the cumulative time delay will converge rather than continue to build in a vacuum-equivalent configuration. It is analogous to a frame dragging phenomenon, but is a zero-mode effect and thus a new "gravitational memory."

Using the constraint equations, one can rewrite this time delay in terms of the angular

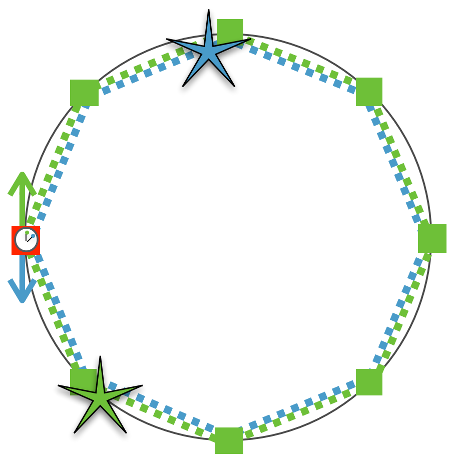


Figure 1: Two counter propagating relays of signals are triggered sequentially around the circular array of detectors, with the final de-sychronization recorded.

momentum aspect and stress tensor:

$$\Delta^{+}u = -\frac{1}{\pi^2 L}\mathrm{Im}\int_{D_{\mathcal{C}}} d^2w\gamma_{w\bar{w}}\int d^2z\partial_{\bar{z}}\mathcal{G}(z;w)\left[\Delta^{+}N_z + \int duT_{uz}\right] \tag{VII.3}$$

where $\mathcal{G}(z;w) = \log(\sin^2\frac{\theta}{2})$. using Andy's boundary matching idea relating antipodal generators on $\mathcal{I}^-$ to $\mathcal{I}^+$, one can consider massless scattering configurations which start and end in vacuum and match

$$\partial_{[z}N_{\bar{z}]}|_{\mathcal{I}^-_+} = \partial_{[z}N_{\bar{z}]}|_{\mathcal{I}^+_-} \tag{VII.4}$$

antipodally across i^0. One then gets

$$\Delta\tau \equiv \Delta^{+}u - \Delta^{-}v = -\frac{1}{\pi^2 L}\mathrm{Im}\int_{D_{\mathcal{C}}} d^2w\gamma_{w\bar{w}}\int d^2z\partial_{\bar{z}}\mathcal{G}(z;w)\left[\int duT_{uz} - \int dvT_{vz}\right] \tag{VII.5}$$

which in terms of the soft factor

$$\frac{1}{2}(\lim_{\omega\to 0+} + \lim_{\omega\to 0-})\mathcal{A}_{n+1}\left(p_1,...p_n;(\omega q,\varepsilon_{\mu\nu})\right) = S^{(1)}_{\mu\nu}\varepsilon^{\mu\nu}\mathcal{A}_n\left(p_1,...p_n\right), \tag{VII.6}$$

where

$$S^{(1)}_{\mu\nu} = \frac{i\kappa}{2}\sum_{k=1}^{n}\frac{p_{k(\mu}J_{k\nu)\lambda}q^{\lambda}}{q\cdot p_k} \tag{VII.7}$$

can be explicitly checked to match for a set of massless scatters. i.e.

$$\mathrm{Im}\Big[\int du D_z^2 C_{\bar{z}\bar{z}} - \int dv D_z^2 C_{\bar{z}\bar{z}}\Big] = 2iG[D_{\bar{z}}^2 \hat{S}_{zz}^{(1)} - D_z^2 \hat{S}_{\bar{z}\bar{z}}^{(1)}]. \tag{VII.8}$$

This boundary matching can be equivalently expressed as a new charge

$$Q(z,\bar{z}) = i\partial_{[\bar{z}} N_{z]}(z,\bar{z})|_{\mathcal{I}^+_-} \tag{VII.9}$$

constraining angular momentum flux through antipodal contours in a scattering process.

As a final note, there is another way to interpret this spin memory in terms of inertial observers. A massive geodesic at large r has a four velocity given by (suppressing the Bondi mass term)

$$v^\mu = (1, 0, -\frac{1}{2r^2} D_z C^{zz}, -\frac{1}{2r^2} D_{\bar{z}} C^{\bar{z}\bar{z}}) \tag{VII.10}$$

The component of the velocity tangental to the two sphere is thus determined by the form $V \equiv \frac{1}{4} D^a C_{ab} dx^b$. If one considers a rigid ring, it can resist deformations but will have no means of resisting a net rotation. Thinking of the curl as the vorticity of a fluid, this time integral is equivalent to a net rotation.

Depending on the relative scaling of the rotation with the size of the contour, it becomes more appropriate or less to treat the ring as a BMS vs. geodesic observer. I.e. for spinning particles passing through a ring the effect scales down rapidly with distance, such that the ring can be "held still," whereas gravitational wave scaling is such that the local region rotates together. While the first gyraton-like solution would expect a time delay, the second would be better described by a net rotation relative to objects far away.

VIII Main Topics

- part that is measureable & part that is a symmetry
  - Δ at boundaries $\leftrightarrow$ supertranslations and an overall shift
  - curl of vector $\int du$ of subleading behavior $\leftrightarrow$ divergence of this vector field's subleading behavior and superrotations
- $u << r$ conformal symmetry of the lightcone
  - $\mathcal{I}^+$ vs stationary observer $t \sim r + \varepsilon$
  - order of limits and u fall-off conditions

- between shocks, looks like pure gauge
  - massless scattering, LSET and the solution propagating out at $u = 0$
- match what happened on $\mathcal{I}^-$ to beginning of what happens on $\mathcal{I}^+$.
  - accelerations occurring at $t = 0$ don't propagate until $u = 0$ to $\mathcal{I}^+$.
  - look at boosted solutions on straight line trajectories through the spacetime origin.
  - antipodal matching of electric charge radial field
- 2 zero modes, Weinberg pole conjugate to $\delta(\omega)$ piece.

Soft Factors

i) E&M Weinberg $\mathcal{O}(\frac{1}{\omega})$
ii) grav. Weinberg $\mathcal{O}(\frac{1}{\omega})$
iii) C.&S. $\mathcal{O}(1)$

Memories

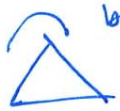
i) L.W.
ii) C.K.
iii) P.S.Z.

Symmetries

	Global	Asymptotic
i)	e-charge	large gauge transformations
ii)	P^{μ}	supertranslations
iii)	$J^{\mu\nu}$	superrotations

$T \times T \times T$

3 copies of this triangle


bridge

Scattering in position space

1) $z = e^{i\phi}\tan\frac{\theta}{2}$ $\quad d\Omega^2 = 2\gamma_{z\bar z}\,dz\,d\bar z$

$\gamma_{z\bar z} = \frac{2}{(1+z\bar z)^2}$

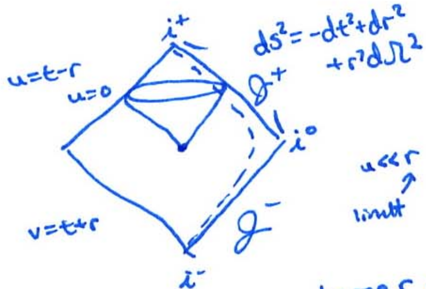
Scattering in momentum space

2) $p^\mu = E(1,\hat x)$ $\quad m=0$

$p^\mu = \frac{m}{\sqrt{1-\beta^2}}(1,\beta\hat x)$ $\quad m\neq 0$


$\hat x$ & E give p

3)



4) $A_\mu(x) = e\sum_{\alpha=\pm}\int \frac{d^3q}{(2\pi)^3}\frac{1}{2\omega_q}\left[\varepsilon_\mu^{\alpha *}a(\vec q)e^{iq\cdot x} + h.c.\right]$

$e^{iq\cdot x} = e^{-i\omega u - i\omega r(1-\hat x\cdot\hat q)}$ ↑ picks out $\hat x = \hat q$

5) larger $\leftrightarrow$ small ω

same $(z,\bar z)$

---

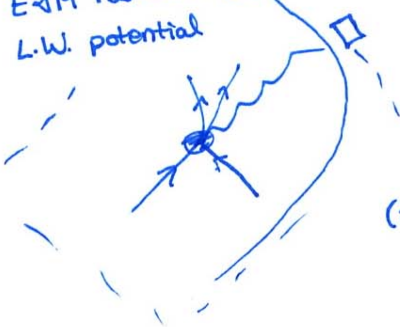
6) $\langle out|a_\alpha S|in\rangle = (\underbrace{S_\alpha^{(0)}}_{\frac{1}{\omega}} + \underbrace{S_\alpha^{(1)}}_{O(1)} + \dots)\langle out|S|in\rangle$

7) $\sim \langle A_\mu\rangle_{\text{given } |in\rangle \to |out\rangle}$ @ $\hat q = \hat x$

what does large r mean?

E&M radiation zone

L.W. potential



8)

Lrg. gauge sym.

2 zero-modes

$\delta(\omega)$ shift

vs $\frac{1}{\omega}$ step

Bondi Mass aspect

Bondi gauge

$$ds^2 = \underbrace{-du^2 - 2du\,dr + 2r^2\gamma_{z\bar z}dz\,d\bar z}_{\eta_{\mu\nu}dx^\mu dx^\nu} + \frac{2m_B}{r}du^2$$
$$+ \left(r C_{zz}dz^2 + D^z C_{zz}\,du\,dz + \frac{1}{r}\left(\frac{4}{3}N_z - \frac{1}{4}\partial_z(C_{zz}C^{zz})\right)du\,dz + c.c.\right)$$

Angular momentum aspect

$\mathcal{L}_\xi g_{\mu\nu}$ preserving this asymptotic form

$$\Rightarrow \xi^+ = \left(1+\frac{u}{2r}\right)Y^z\partial_z - \frac{u}{2r}D^{\bar z}D_z Y^z\partial_{\bar z} - \frac{1}{2}(u+r)D_z Y^z\partial_r + \frac{u}{2}D_z Y^z\partial_u + c.c.$$

CKV on S^2 "superrotations"

$$+ f\partial_u - \frac{1}{r}\left(D^z f\partial_z + D^{\bar z} f\partial_{\bar z}\right) + D^z D_z f^+\partial_r$$

fn on S^2 "supertranslations"

Solving $G_{\mu\nu} = 8\pi G T^M_{\mu\nu}$

$$\Rightarrow \partial_u m_B = \frac{1}{4}\partial_u\left[D_z^2 C^{zz} + D_{\bar z}^2 C^{\bar z\bar z}\right] - T_{uu}$$
$$\partial_z\partial_{\bar z} m_B = \mathrm{Re}\left[\partial_u\partial_{\bar z}N_z + \partial_{\bar z}T_{uz}\right]$$
$$\mathrm{Im}\left[\partial_{\bar z}D_z^3 C^{zz}\right] = 2\,\mathrm{Im}\left[\partial_u\partial_{\bar z}N_z + \partial_{\bar z}T_{uz}\right]$$

where I have absorbed quadratic terms into $T_{\mu\nu}$ vs. $T^M_{\mu\nu}$

Route to $\langle Q^+S - SQ^-\rangle = 0$

- b.c. and $\mathcal{I}^-$ to $\mathcal{I}^+$ matching
- $\mathcal{L}_\xi$ on hard particles (Q_H) $\Leftrightarrow$ soft factor Q_S part
- $\delta_Y C_{zz} = -uD_z^3 Y^z + \ldots \rightarrow Q_S^+ = -\frac{1}{2}\int du\,d^2z\,D_z^3 Y^z u\partial_u C^{\bar z}{}_{\bar z}$ ← $\int du \rightarrow \mathcal{O}(1)$
- $\delta_f C_{zz} = -2D_z^2 f + \ldots$ s.grav. current $\sim \int du\,\partial_u C_{zz}$

$\rightarrow \frac{1}{\omega}$

Standard C.K. grav. mem. uses

G_{uu} constraint $\Rightarrow C_{zz} = -2D_z^2 C$ @ late & early times

similar to "pure gauge" between shocks / fronts

s.p. $\Leftrightarrow$ boosted m_B solution

What about subleading soft theorem?

found s.rot. charge using $S^{(1)}$ since $\int du\, u\partial_u \leftrightarrow \lim_{\omega\to 0}(1+\omega\partial_\omega)$

want a time integrated effect

$\Rightarrow$ closed path?

$\Rightarrow$ light on forced circle $ds^2=0$

$\uparrow$ vs $\downarrow$ pick up opposite $du\,dz$ term

$\Rightarrow$ over 1 cycle $\Delta P = \oint_c (D^z C_{zz}\, dz + D^{\bar z} C_{\bar z\bar z}\, d\bar z)$

$\Rightarrow$ each cycle adds!

slowly varying $\Leftrightarrow$ sampling every $2\pi L$

$\Delta u^+ = \frac{1}{2\pi L}\int du\, \Delta P(u)$

$\Rightarrow$ use 2nd G_{uz} constraint and $\partial_{[z}N_{\bar z]}\big|_{I^+} = \partial_{[z}N_{\bar z]}\big|_{I^-}$ antipodal matching

$Q(z,\bar z) = i\partial_{[\bar z}N_{z]}(z,\bar z)\big|_{I^\pm}$ relates ang. mom. flux on pairs of contours

$\Delta^+ u - \Delta^- v = -\frac{1}{\pi^2 L}\,\mathrm{Im}\int_{D_c} d^2w\, \gamma_{w\bar w}\int d^2 z\, \partial_{\bar z} G(z;w)\left[\int du\, T_{uz} - \int dv\, T_{vz}\right]$

$\uparrow \ln\left|\frac{(z-w)^2}{(1+w\bar w)(1+z\bar z)}\right|$

$\Rightarrow$ Agrees with subleading soft factor

$\frac{1}{2}\left(\lim_{\omega\to0^+}+\lim_{\omega\to0^-}\right)A_{n+1}(p_i,(\omega q,\epsilon_{\mu\nu})) = S^{(1)}_{\mu\nu}\epsilon^{\mu\nu}A_n(p_i)$

where $S^{(1)}_{\mu\nu} = \frac{i\kappa}{2}\sum_k \frac{p_{k\mu}J_{k\nu\lambda}q^\lambda}{q\cdot p_k}$

$\mathrm{Im}\left[\int du\, D_z^2 C_{\bar z\bar z} - \int dv\, D_z^2 C_{\bar z \bar z}\right] = 2iG\left[D_{\bar z}^2 \hat S^{(1)}_{zz} - D_z^2\hat S^{(1)}_{\bar z\bar z}\right]$ ✓

$\Rightarrow$ Final note geodesic w/ $m_B\to 0$ has

$v^\mu = (1,0,-\frac{1}{2r^2}D_z C^{zz}, -\frac{1}{2r^2}D_{\bar z}C^{\bar z\bar z})$

$\uparrow$ velocity tangent to S^2

rigid ring does not resist rotation

$\Rightarrow$ time integral gives a net rotation!

Speed Demon

Sabrina Gonzalez Pasterski
(Dated: Dec 16, 2014)

This note proposes a detector arrangement/measurement corresponding to the subleading soft graviton theorem.[a]

I. STARTING ASSUMPTIONS

The metric conditions considered in S.&Z.[1] were:

$$\begin{array}{ll} m_b = M_i = constant, & C_{zz} = 0 \\ m_b = M_f = constant, & C_{zz} \neq 0. \end{array} \tag{I.1}$$

Here, I will consider a particular scattering configuration where instead $C_{zz} = 0$ both initially and finally. Moreover, I will restrict myself to situations where the envelope of $C_{zz}(u)$ has a finite u integral at each point on the sphere. Under these conditions:

$$\begin{aligned} \int duu\partial_u C_{zz} &= uC_{zz}|^{\infty}_{-\infty} - \int duC_{zz} \\ &= -\int duC_{zz} \end{aligned} \tag{I.2}$$

where the boundary term can be dropped for quick enough $C_{zz}(u)$ fall-offs, which I will assume.

II. TWO DETECTOR PRIMER

Following S.&Z., I will consider detectors that are at a fixed $r = r_0$, with $\delta z = z' - z$ describing their angular separation in complex coordinates. If we assume δz is small, then:

$$L = \frac{2r_0|\delta z|}{1 + z\bar{z}} \tag{II.1}$$

is their spatial separation using the standard flat metric:

$$ds_F^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}. \tag{II.2}$$

When, in addition to the flat metric, there is a perturbation:

$$\begin{aligned} ds^2 = \ & ds_F^2 + \tfrac{2m_B}{r}du^2 \\ & + rC_{zz}dz^2 + D^zC_{zz}dudz + c.c., \end{aligned} \tag{II.3}$$

the trajectories of light rays traveling between detectors will satisfy:

$$2r_0^2\gamma_{z\bar{z}}\delta z\delta\bar{z} + r_0C_{zz}\delta z^2 + D^zC_{zz}\delta_{zz'}u\delta z + c.c. - (\delta_{zz'}u)^2 = 0 \tag{II.4}$$

going from $z \to z'$, whereas the reverse route will have:

$$2r_0^2\gamma_{z\bar{z}}\delta z\delta\bar{z} + r_0C_{zz}\delta z^2 - D^zC_{zz}\delta_{z'z}u\delta z + c.c. - (\delta_{z'z}u)^2 = 0 \tag{II.5}$$

where the Bondi mass term is subleading in r_0. Subtracting the two equations gives:

$$\delta_{zz'}u - \delta_{z'z}u = D^zC_{zz}\delta z + c.c. \tag{II.6}$$

While adding the two equations gives:

$$(\delta_{zz'}u)^2 + (\delta_{z'z}u)^2 = 4r_0^2\gamma_{z\bar{z}}\delta z\delta\bar{z} + r_0C_{zz}\delta z^2 + c.c. \tag{II.7}$$

Combing these yields:

$$\delta_{zz'}u = \tilde{L} + \frac{1}{2}[D^zC_{zz}\delta z + c.c.] \tag{II.8}$$

$$\delta_{z'z}u = \tilde{L} - \frac{1}{2}[D^zC_{zz}\delta z + c.c.] \tag{II.9}$$

where

$$\tilde{L} = L + \frac{r_0}{2L}[C_{zz}\delta z^2 + c.c.] \tag{II.10}$$

III. SUBLEADING MEASUREMENT

Consider N detectors arranged in a regular polygon around z=0.

$$z_n = \epsilon e^{i\frac{2\pi n}{N}}, \quad n \in \{0, ..., N-1\} \tag{III.1}$$

In the large N limit, one has:

$$z = \epsilon e^{i\phi}, \quad \delta z = iz\delta\phi \tag{III.2}$$

The difference between a clockwise versus a counter clockwise circuit for a constant C_{zz} is:

$$\begin{aligned} &\lim_{N\to\infty}\sum_{n=0}^{N-1}\{\delta_{n,n+1}u - \delta_{n+1,n}u\} \\ &= \lim_{N\to\infty}\sum_{n=0}^{N-1} D^zC_{zz}(\epsilon e^{i\frac{2\pi n}{N}})i\epsilon e^{i\frac{2\pi n}{N}}\frac{2\pi}{N} + c.c. \\ &= \int_0^{2\pi} D^zC_{zz}(z)iz\delta\phi + c.c. \\ &= \oint_\epsilon D^zC_{zz}dz + c.c. \end{aligned} \tag{III.3}$$

This is equivalent to sending the signal chains in opposite directions around the array of detectors and looking at the time difference when the two pulses arrive at the starting point after a single loop.

Note that this is a cumulative effect. The difference in timing is a correction to the net time for a single circuit, which is at leading order in r_0 is:

$$L_\epsilon = 4\pi r_0\epsilon. \tag{III.4}$$

---

[a] ISBN: 978-0-9863685-9-2

[1] arXiv:1411.5745v1.pdf

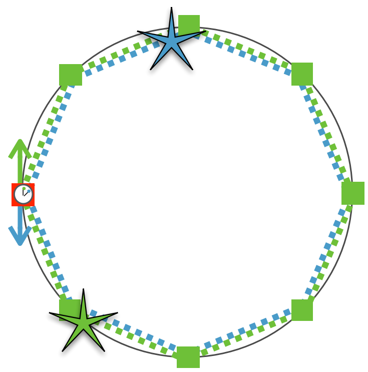


FIG. 1. Two counter propagating relays of signals are triggered sequentially around the circular array of detectors, with the final de-sychronization recorded.

The subleading correction, which does not change the relative time delay of the counter rotating signals, is due to the difference between the loop integrals of L and $\tilde{L}$.

Consider a $C_{zz}(u)$ which varies slowly enough that summing the accumulated time delay for intervals spaced by $\Delta u = 4\pi r_0\epsilon$ approximates the u integral:

$$\begin{aligned}&\sum_m \oint_\epsilon D^z C_{zz}(u = 4\pi r_0 \epsilon m) dz + c.c. \\ &= \tfrac{1}{4\pi r_0 \epsilon} \int du \oint_\epsilon D^z C_{zz} dz + c.c.\end{aligned} \tag{III.5}$$

Under the assumption that the u integral is finite, the final time delay between two initially synched counter rotating signal chains thus corresponds to u, z integral.

IV. SOFT FACTOR INTERPRETATION

The net de-synchronization is related to the subleading soft graviton factor. For a stationary metric satisfying the the vacuum Einstein equations, it would be the time integral of the curl of the twist, which would be zero. In a scattering process with gravitational radiation, however, one can use the expectation value interpretation to think of $\int du D_{\bar{z}}^2 C_{zz}$ in terms of the soft factor. In this case, the real and imaginary parts of the subleading soft factor correspond to divergence and curl of $D^a C_{ab}$. Note that the subleading soft graviton theorem contains more content than that used for the superrotation charge. The real part of $D_{\bar{z}}^2 C_{zz}$ is what appears in the superrotation charge, $\int du D^a D^b C_{ab}$, while the imaginary part contributes to $i\int du[D_{\bar{z}}^2 C_{zz} - D_z^2 C_{\bar{z}\bar{z}}]$. This imaginary part is expected to be finite under the boundary conditions[2] $D_z^2 C_{\bar{z}\bar{z}} - D_{\bar{z}}^2 C_{zz} = 0$ at $\mathcal{I}_+^+$ and $\mathcal{I}_-^+$, assuming quick enough fall offs in u for this quantity.

ACKNOWLEDGEMENTS

Many thanks to A. Strominger and A. Zhiboedov for useful questions.

[2] arXiv:1406.3312.pdf