Classical Interpretation of the Weinberg Soft Factor

Sabrina Gonzalez Pasterski

(Dated: July 31, 2014)

I show how the radiation emitted during the scattering of non-relativistic charged particles corresponds to the $O(\omega^{-1})$ soft factor in QED. Namely, if we think of the photon momentum in the soft factor as labeling a direction at which a far-field observer sits, the QED matrix element pre-factor corresponds to the time integral of the radiated electric field measured by that observer when a set of non-relativistic charged particles scatter and accelerate.

I. CLASSICAL SCATTERING

Consider the mode expansion of $F_{uz} = \partial_u A_z$ from “Low’s Subleading Soft Theorem as a Symmetry of QED”:

$$F_{uz} = - \frac{e^2}{8\pi^2} \int_0^\infty \omega a_+ (\omega \bar{v}) e^{-i\omega u} + a_- (\omega \bar{v}) e^{i\omega u}]$$.

(II.1)

Its integral over $u$ is given by:

$$\int du F_{uz} = - \frac{e^2}{8\pi} \lim_{\omega \to 0^+} \omega [a_+ (\omega \bar{v}) + a_- (\omega \bar{v})^\dagger]$$.

(II.2)

so that a soft insertion picks out: $\omega \bar{v}_\perp$ times the Weinberg soft factor.

Semi-classically, we can think of the mode expansion of $F_{uz}$ as the Fourier transform for the corresponding electric field component. The Weinberg soft theorem thus corresponds to the time integral of the radiated electric field measured at any far-field point labelled by $(z, \bar{z})$. Such a non-zero time-integrated value would be expected for a charged particle that accelerates.

Let’s look at some equations from classical electrodynamics:

$$\vec{E}^{\text{rad}} = \vec{e}_r \times \left( \vec{e}_r \times \frac{\partial \vec{A}^{\text{rad}}}{\partial t} \right)$$.

(II.3)

where $\frac{\partial \vec{A}^{\text{rad}}}{\partial t}$ becomes the $u$ derivative of the gauge field component tangent to the two sphere at the far-field point, just as $F_{uz}$ is the $u$ derivative of $A_z$. For a non-relativistic accelerating particle:

$$\vec{E}_{\text{rad}} = \frac{Q}{4\pi \epsilon_0 r^2} \vec{e}_r \times (\vec{e}_r \times \vec{a})$$.

(II.4)

so that the time integral of $\vec{E}_{\text{rad}}$ is proportional to the change in velocity of the particle. For instance, in the non-relativistic regime where the same particles come in and out, but with different velocities:

$$\int dt \vec{E}_{\text{rad}} = \sum_{\text{in} - \text{out}} \frac{Q k}{4\pi \epsilon_0 r c^2} \vec{e}_r \times (\vec{e}_r \times \vec{v}_k)$$.

(II.5)

II. CONNECTION TO QED SOFT FACTOR

For a far-field point labeled by $(z, \bar{z})$, we have:

$$\vec{e}_r = \left( \frac{z + \bar{z}}{1 + z \bar{z}}, \frac{i(\bar{z}z - 1)}{1 + z \bar{z}} \right)$$.

(II.6)

while a particle traveling with four momentum:

$$p_k = |p_k| \left( \frac{1 + m_k^2}{|p_k|^2}, \frac{z_k + \bar{z}_k}{1 + z_k \bar{z}_k}, \frac{i(z_k - \bar{z}_k)}{1 + z_k \bar{z}_k}, \frac{1 - z_k \bar{z}_k}{1 + z_k \bar{z}_k} \right)$$.

(II.7)

has, at leading order in the non-relativistic limit:

$$\vec{v}_k = \frac{|p_k|}{m_k} \left( \frac{z_k + \bar{z}_k}{1 + z_k \bar{z}_k}, \frac{i(z_k - \bar{z}_k)}{1 + z_k \bar{z}_k}, \frac{1 - z_k \bar{z}_k}{1 + z_k \bar{z}_k} \right)$$.

(II.8)

We then find that

$$\sum_{\text{in} - \text{out}} \frac{Q_k}{|p_k|} \cdot \epsilon^+ \cdot \vec{v}_k \cdot \vec{a} = \sum_{\text{in} - \text{out}} -2 \frac{Q_k |p_k|}{m_k r} \left( (z_k - \bar{z}_k)(1 + z_k \bar{z}_k) \right)$$.

(II.9)

where $\vec{a} = r \vec{e}_r$.

Meanwhile, in the low-particle-momentum limit

$$\sum_{\text{in} - \text{out}} \omega k \cdot \epsilon^+ \cdot \epsilon^- \cdot q = \sum_{\text{in} - \text{out}} -2 \frac{Q_k |p_k|}{m_k r} \left( (z_k - \bar{z}_k)(1 + z_k \bar{z}_k) \right)$$.

(II.10)

for photon momentum and polarization four vectors given by:

$$q = \omega \left( 1, \frac{z + \bar{z}}{1 + z \bar{z}}, \frac{i(\bar{z}z - 1)}{1 + z \bar{z}} \right)$$.

(II.11)

$$\epsilon^+ = \frac{1}{\sqrt{2}} (\bar{z}, 1, -i, -\bar{z})$$.

(II.12)

Similarly,

$$\sum_{\text{in} - \text{out}} \left\{ \frac{Q_k}{r} \cdot \epsilon^- \times (\epsilon^- \times \vec{v}_k) \right\} \cdot \vec{a} = \sum_{\text{in} - \text{out}} \omega k \cdot \epsilon^- \cdot \epsilon^-$$.

(II.13)

where $\epsilon^- = \epsilon_\perp^+$ in Minkowski coordinates.

We thus see that the Weinberg soft factor appearing in the mode expansion of $F_{uz}$ in the $\omega \to 0$ limit corresponds to the total time integral of the electric field radiating towards $(z, \bar{z})$ coming from the acceleration of massive charged particles when their velocities change in a scattering process.

ACKNOWLEDGEMENTS

Many thanks to J. Barandes and A. Strominger.