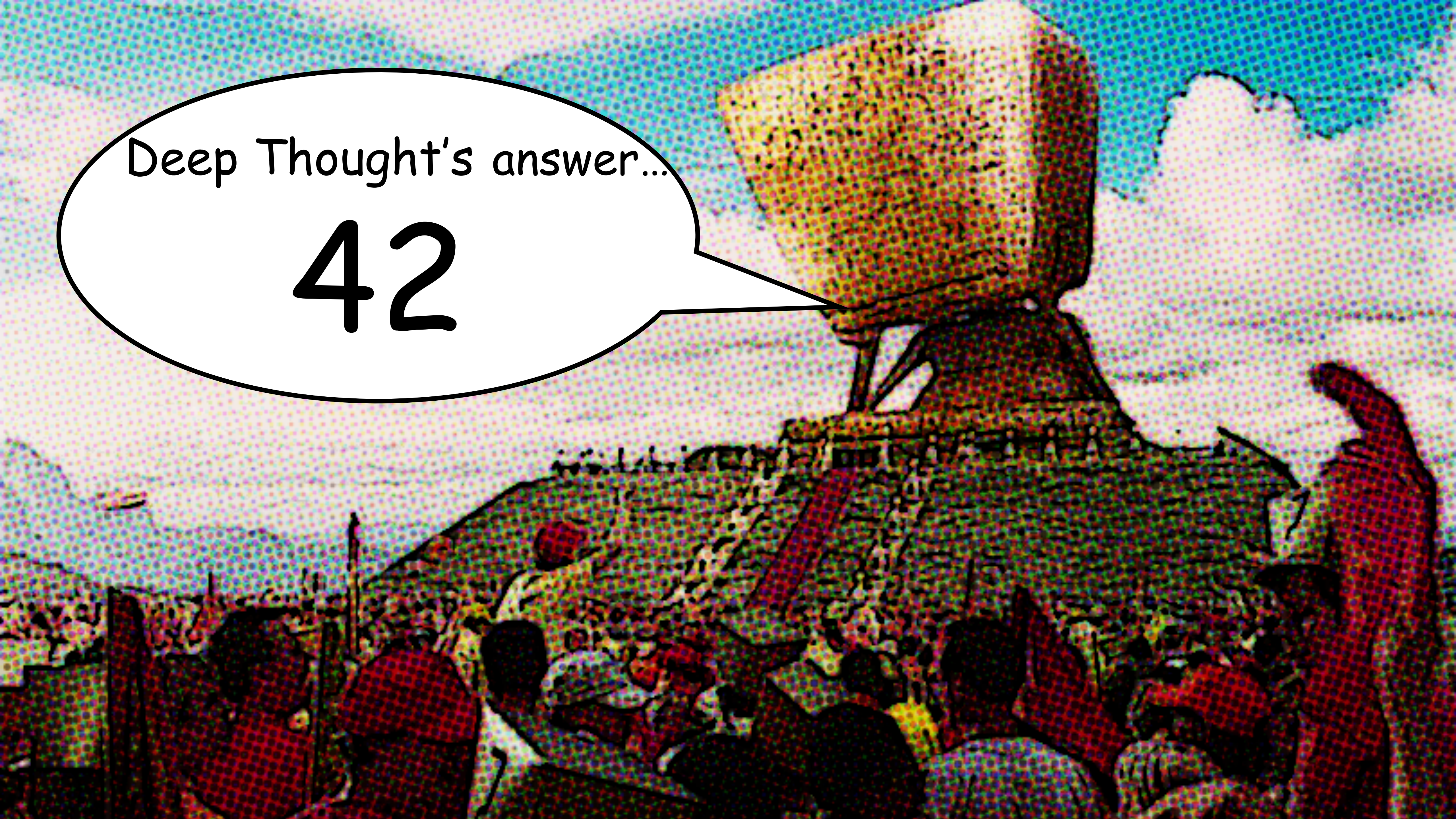
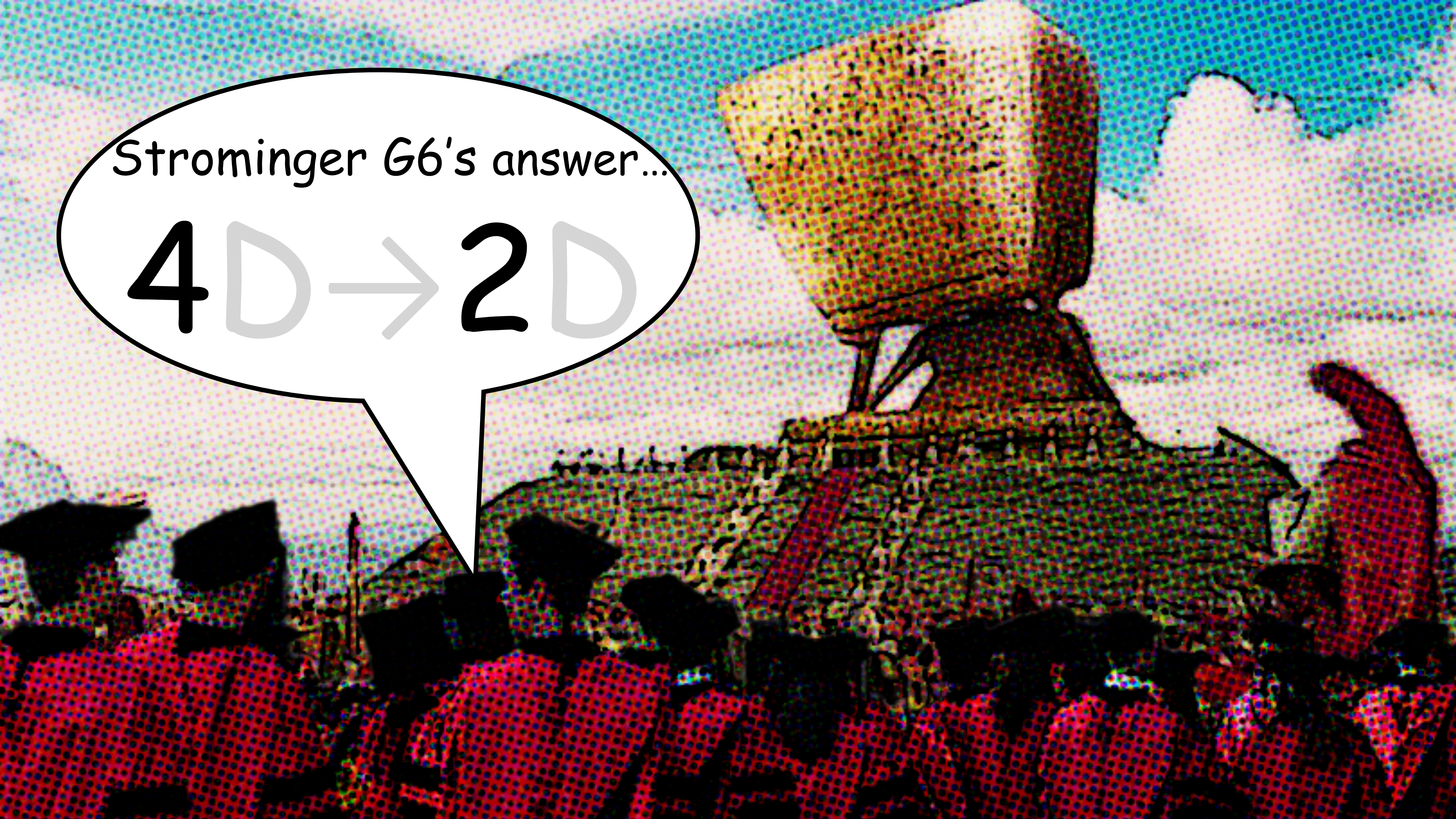
What a G1 hopes to find…

“the ultimate answer to life,
the universe and everything…”


Deep Thought's answer…
42

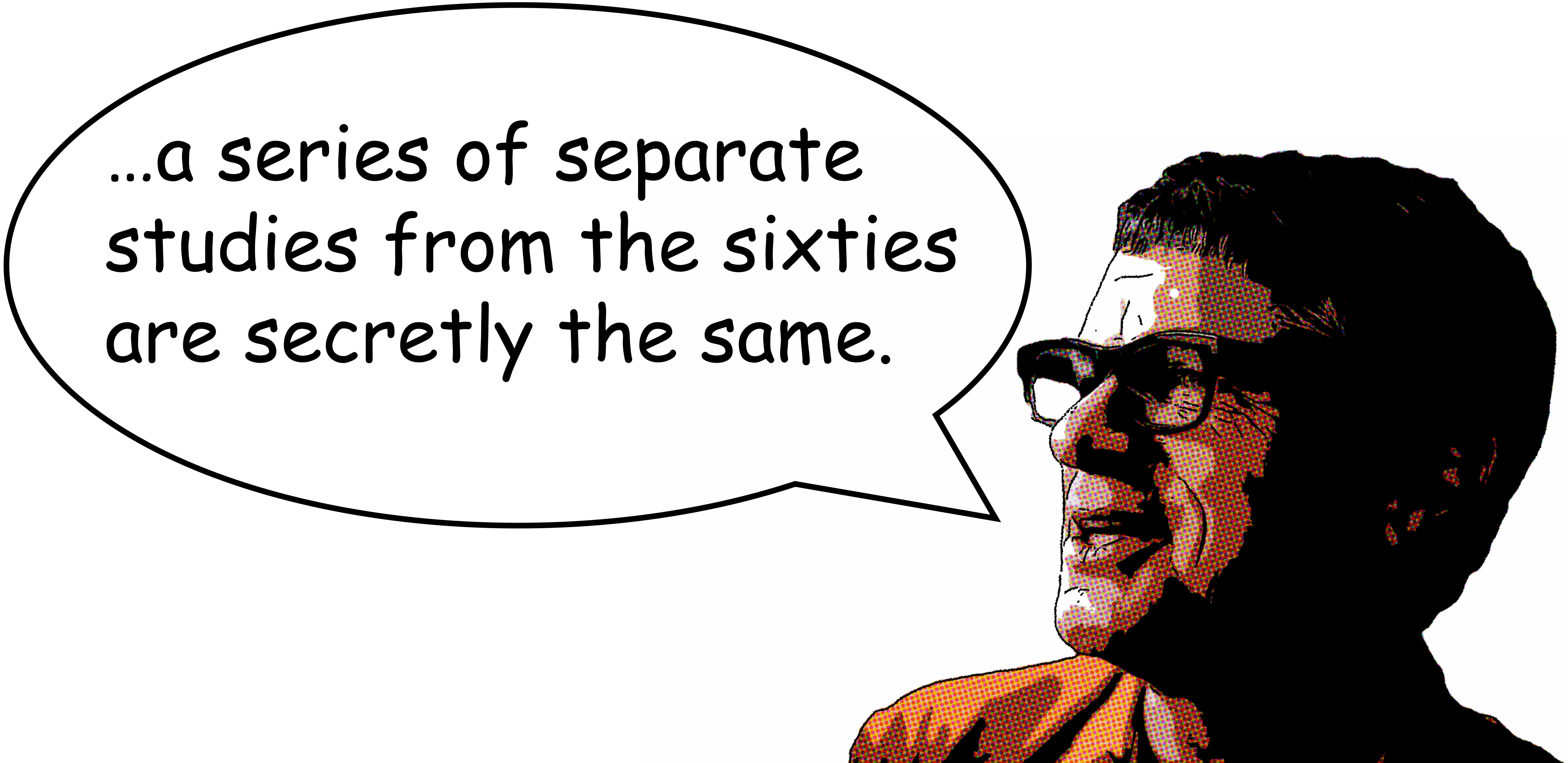

Strominger G6's answer…
4D→2D

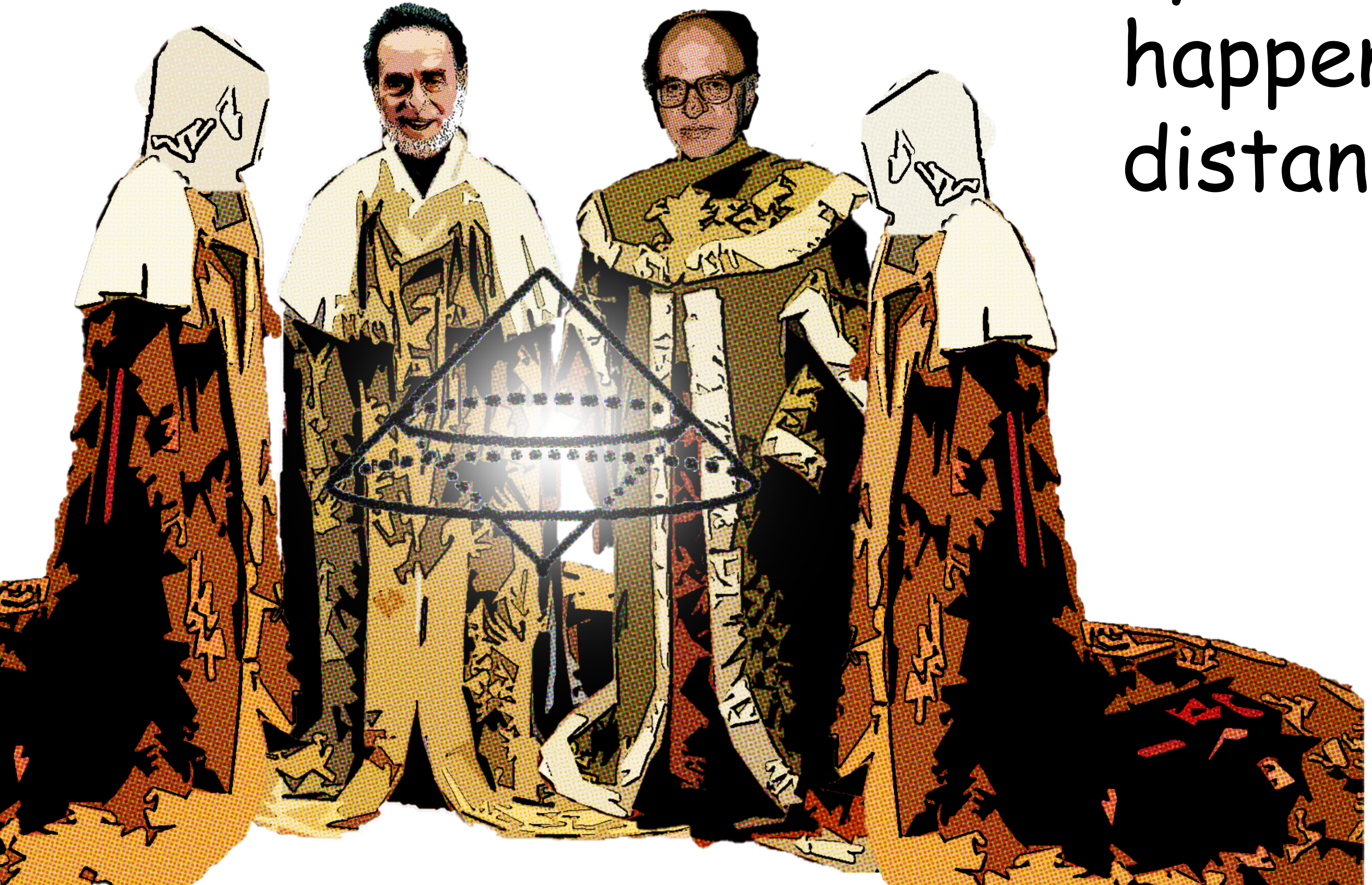
Implications of Superrotations

Sabrina Pasterski

04/18/19

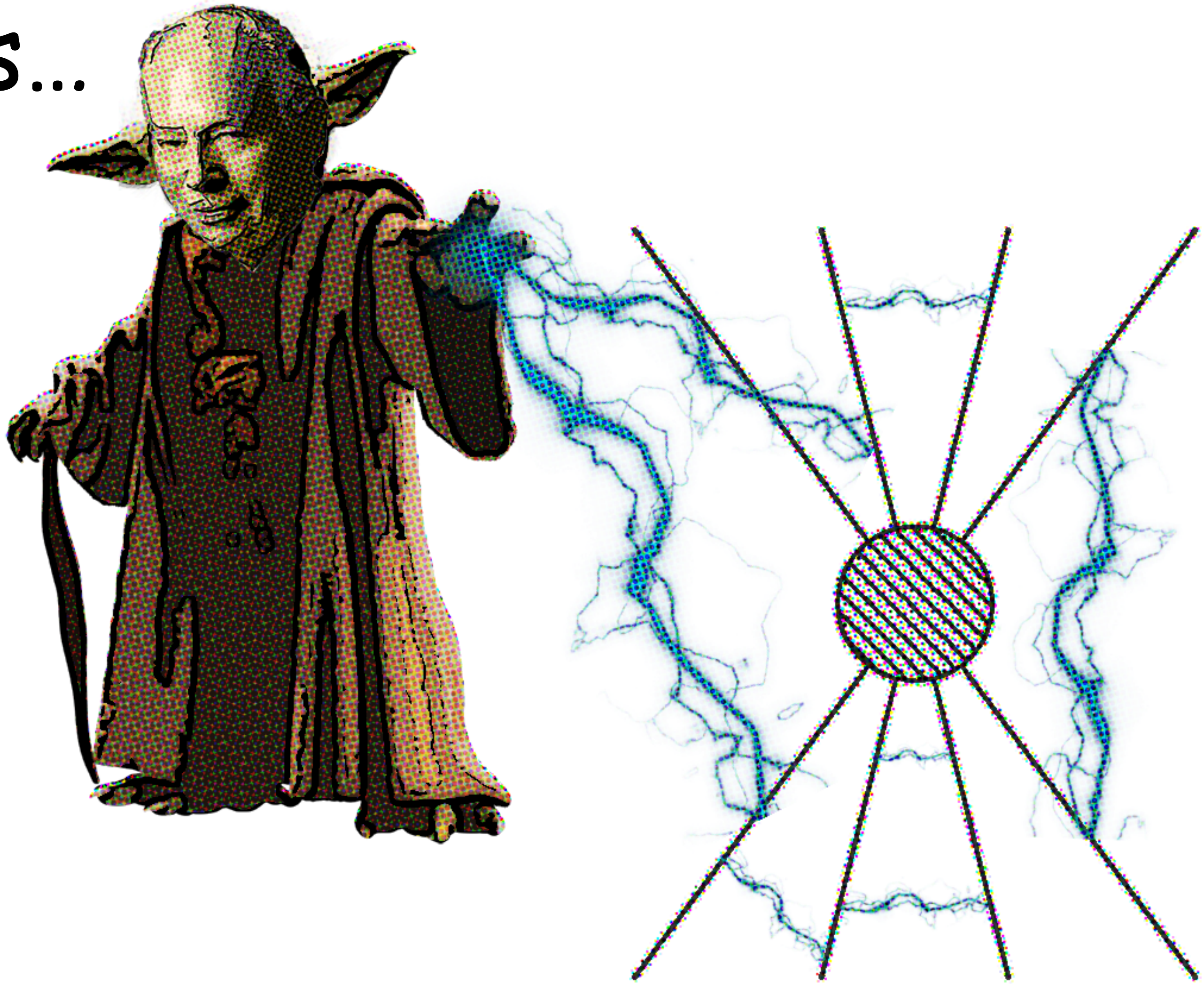
Our story starts with Strominger's suggestion that...


…a series of separate studies from the sixties are secretly the same.



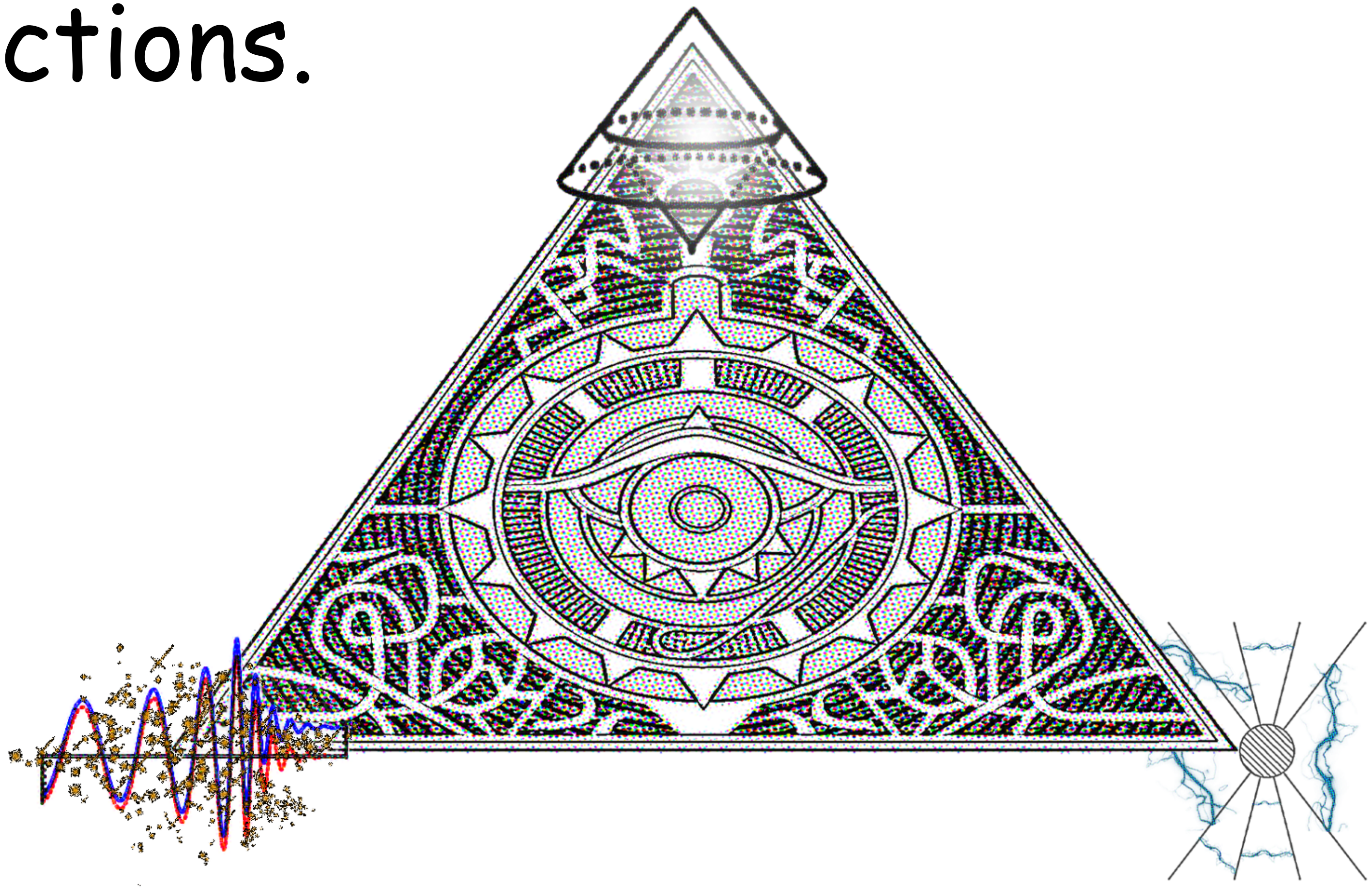
The relativists were systematizing what happens at long distances…

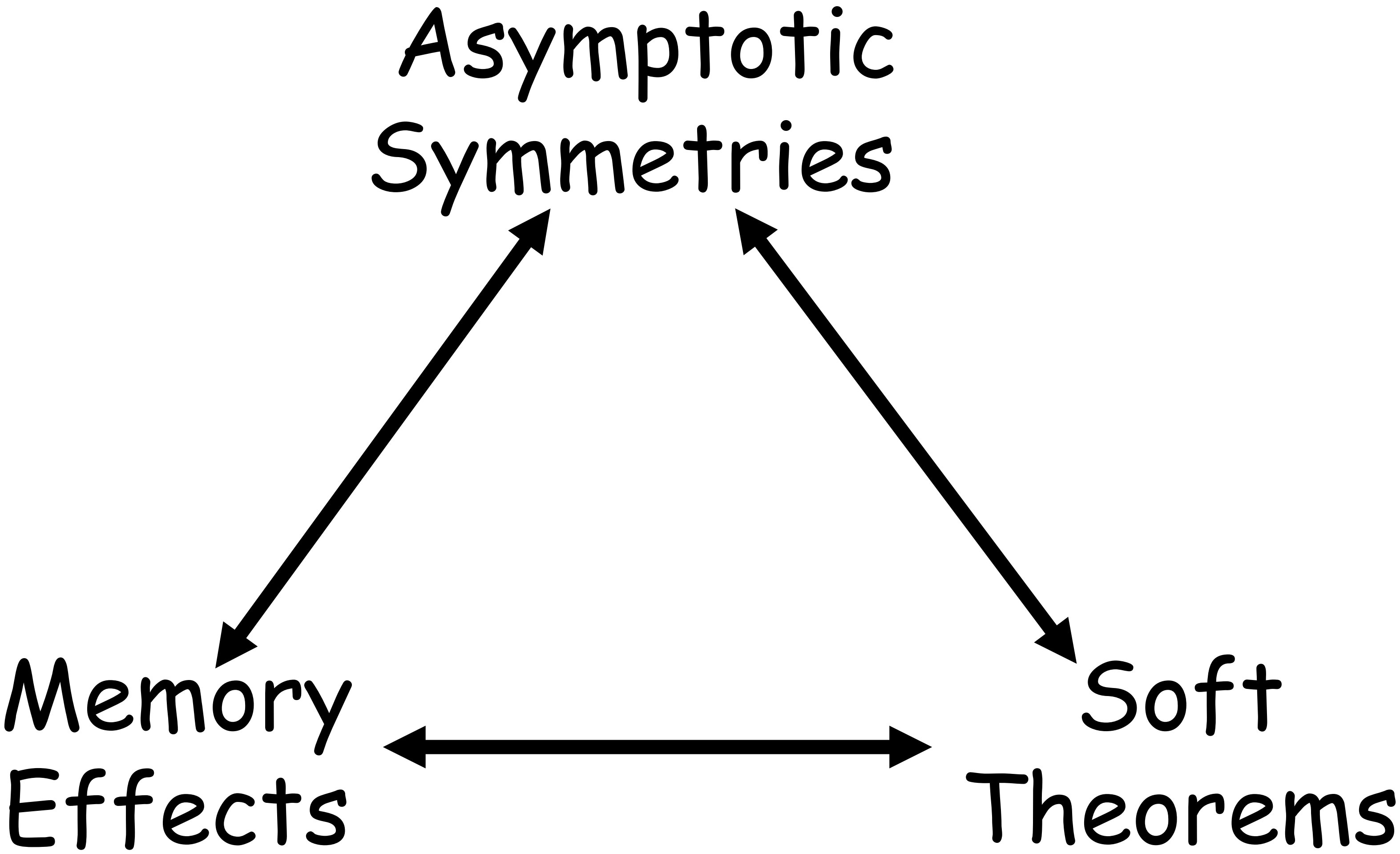
The quantum field theorists were worried about what was going on at low energies…



And, a little later, someone remembered there was a physical observable attached to each of these things….

Together they formed a triangle of traits universal enough to make new predictions.





1505.00716

In QFT, soft theorems imply…

an insertion of a soft gauge field operator	=	a function of charged *external* states only

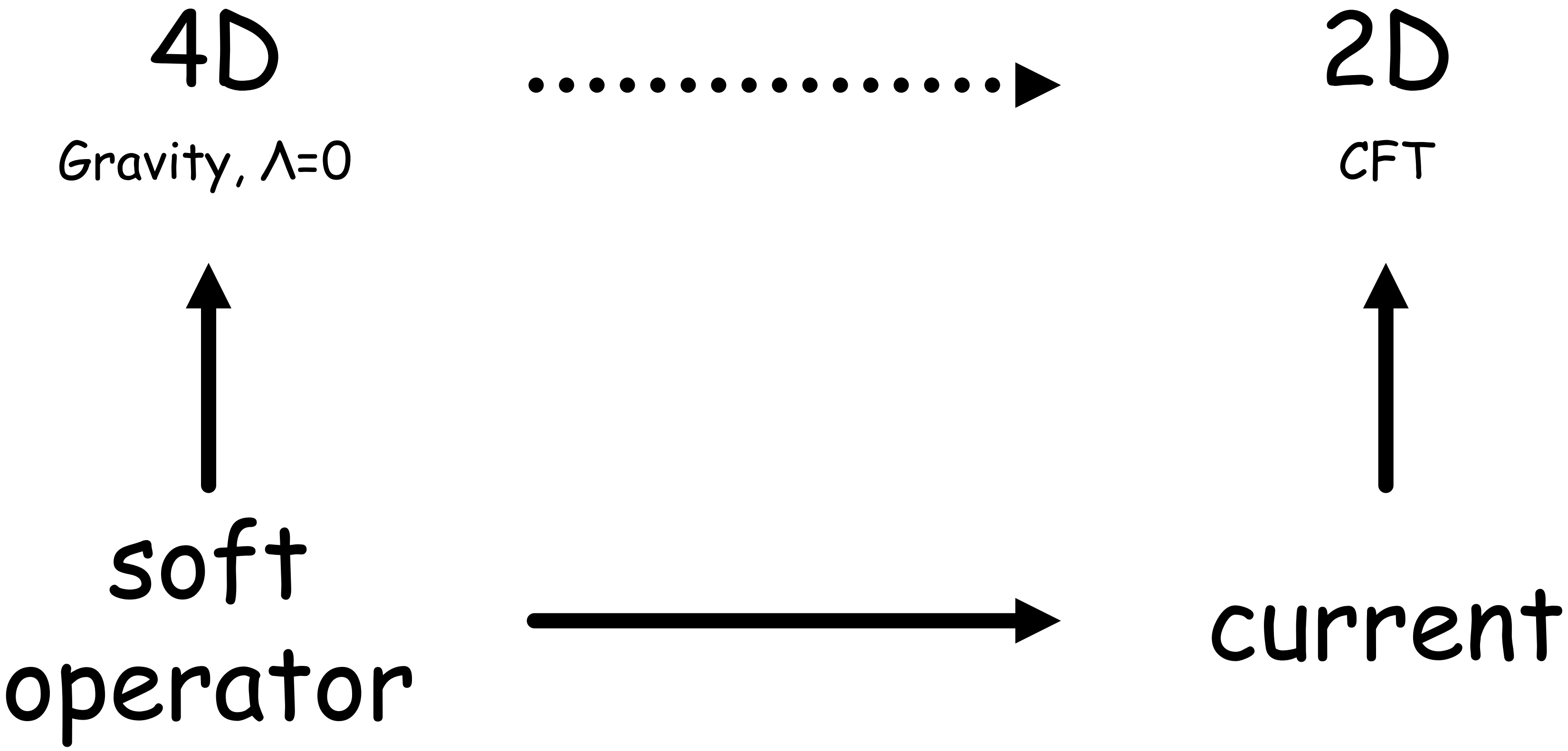
From the perspective of asymptotic symmetries…

$$Q^{+} = Q^{-} \quad \& \quad Q^{\pm} = Q_S^{\pm} + Q_H^{\pm}$$

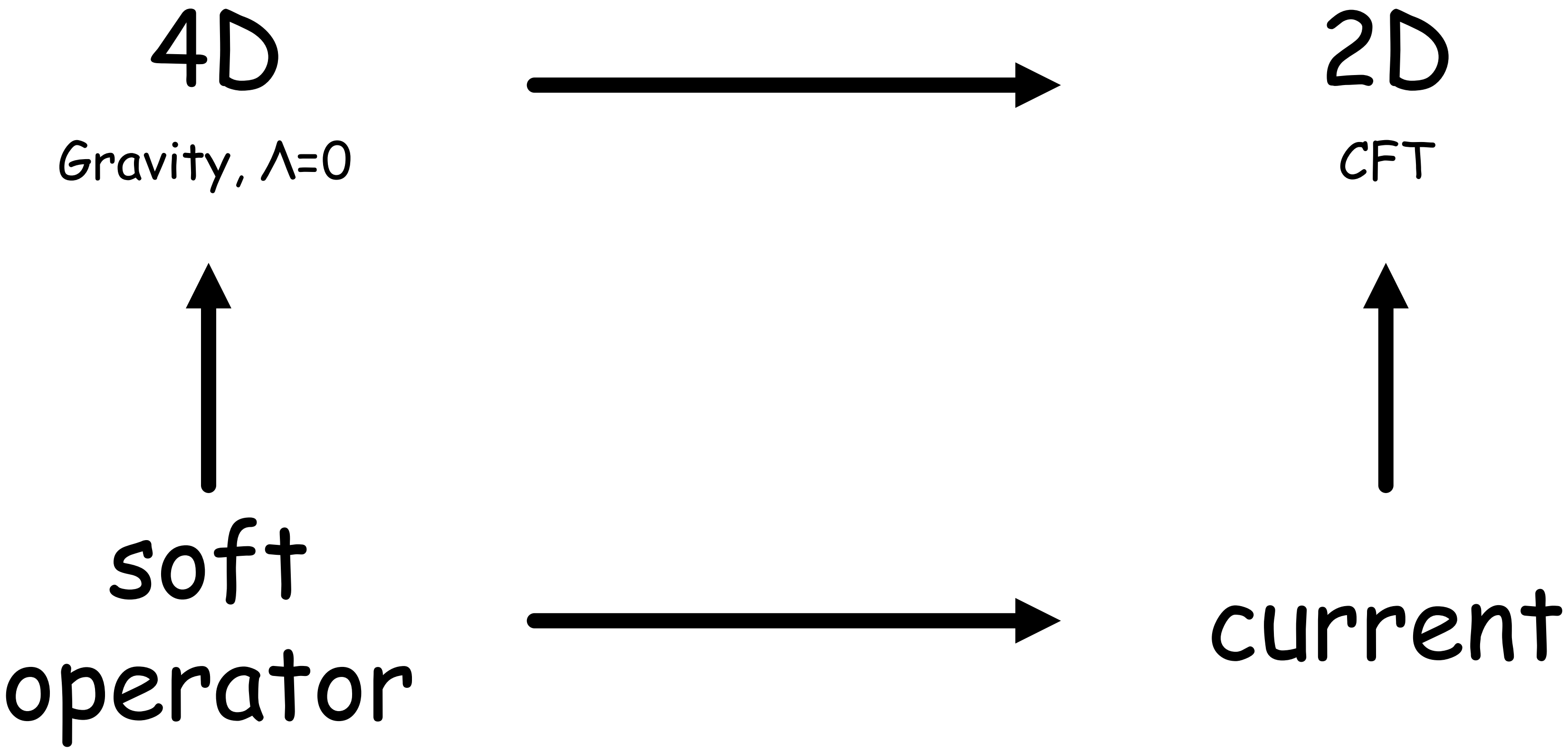
… soft theorems are a manifestation of ***charge conservation***.

The soft operator Q_S has a semi-classical interpretation as a *low energy observable* called a memory effect…

… and appears to have a 2D interpretation as a current.


4D
Gravity, Λ=0
2D
CFT
soft
operator
current

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{c}\text{4D}\\ \text{Gravity, }\Lambda=0\end{array} \arrow[r, dotted] & \begin{array}{c}\text{2D}\\ \text{CFT}\end{array} \\
\begin{array}{c}\text{soft}\\ \text{operator}\end{array} \arrow[u, "\begin{array}{c}\text{Ward identity}\\ \text{asym. sym. grp.}\end{array}"] \arrow[r] & \text{current} \arrow[u, swap, "\begin{array}{c}\text{Ward identity}\\ \text{holo. current}\end{array}"]
\end{tikzcd}
$$


4D
Gravity, Λ=0
2D
CFT
soft operator
current

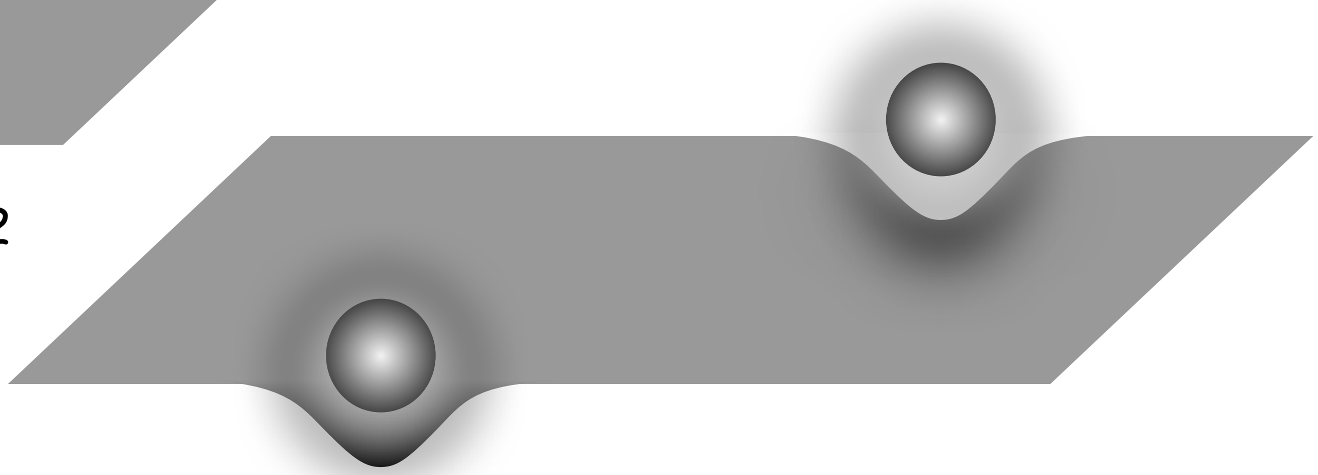
Note we haven't specified to any particular asymptotic symmetry yet, hence the beauty of the IR Triangle…

… Here we care about superrotations

- subleading soft grav. thm.
- spin memory
- stress tensor

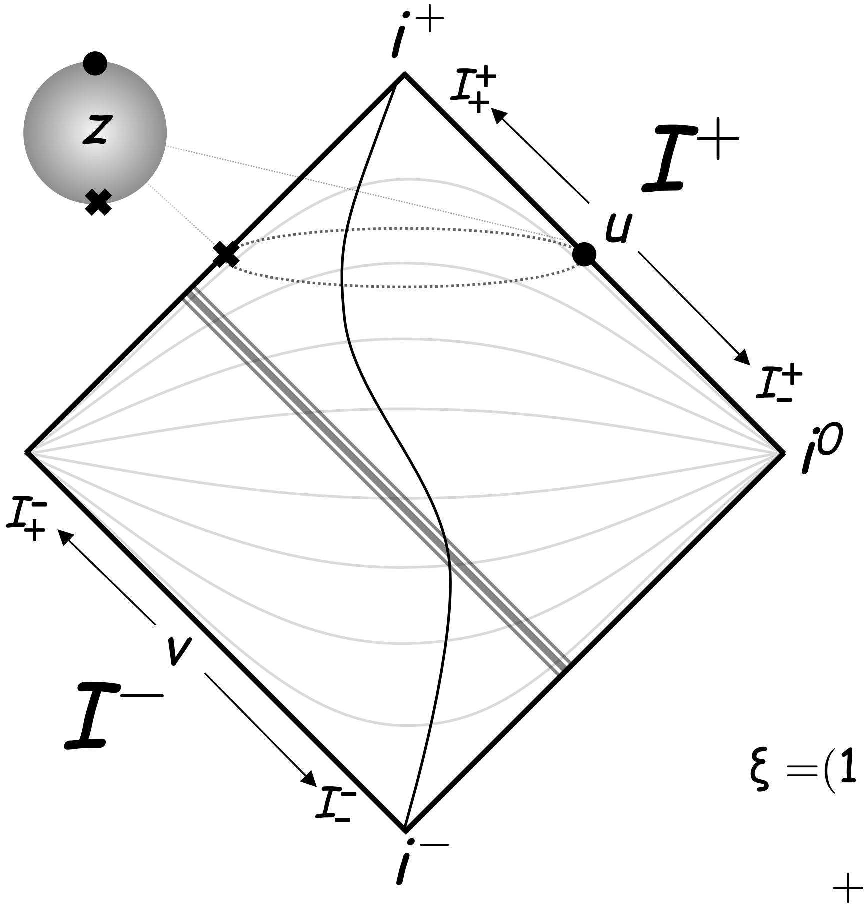
We are interested in the class of asymptotically flat spacetimes…

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$



… with localized stress tensor sources.

The large-r boundary behavior…



$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} + \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 + \left[(D^zC_{zz} - \frac{1}{4r}\partial_z(C_{zz}C^{zz}) + \frac{4}{3r}N_z)dudz + \text{c.c}\right] + \ldots$$

… is preserved by the diffeomorphisms…

$$\xi = (1+\frac{u}{2r})Y^z\partial_z - \frac{u}{2r}D^{\bar{z}}D_zY^z\partial_{\bar{z}} - \frac{1}{2}(u+r)D_zY^z\partial_r + \frac{u}{2}D_zY^z\partial_u + \text{c.c.} + f\partial_u - \frac{1}{r}(D^zf\partial_z + D^{\bar{z}}f\partial_{\bar{z}}) + D^zD_zf\partial_r + \ldots$$

1406.3312

… where superrotations are parameterized by $Y^z(z)$ and have non-zero charge…

$$Q^+(Y) = \frac{1}{8\pi G}\int_{I^+_-} \sqrt{\gamma} d^2z \left(u D_A Y^A m_B + Y^A N_A \right)$$

… which would make them part of the asymptotic symmetry group.

$$ASG = \frac{\text{Allowed Gauge Symmetries}}{\text{Trivial Gauge Symmetries}}$$

1406.3312

Using the constraint equations…

$$\partial_u m_B = \frac{1}{4}\left[D_z^2 N^{zz} + D_{\bar z}^2 N^{\bar z\bar z}\right] - T_{uu}$$

$$\partial_u N_z = \frac{1}{4}\partial_z\left[D_z^2 C^{zz} - D_{\bar z}^2 C^{\bar z\bar z}\right] + \partial_z m_B - T_{uz}$$

… the charge can be written as an integral along null infinity…

$$Q^+(Y) = \frac{1}{8\pi G}\int_{\mathcal{I}^+} \sqrt{\gamma} d^2z du \left[-\frac{1}{2}D_z^3 Y^z u\partial_u C^{zz} + Y^z T_{uz} + u D_z Y^z T_{uu} + \text{h.c.}\right]$$

1406.3312

… which breaks into a soft and a hard part…

$$Q^+(Y) = \frac{1}{8\pi G}\int_{I^+}\sqrt{\gamma}d^2z du\left[\underbrace{-\frac{1}{2}D_z^3 Y^z u\partial_u C^{zz}}_{Q_S^+} + \underbrace{Y^z T_{uz} + uD_z Y^z T_{uu}}_{Q_H^+} + h.c.\right]$$

… where the soft part can be understood in terms of the mode expansion…

$$C_{\bar{z}\bar{z}} = -\frac{i\kappa}{8\pi^2}\hat{\varepsilon}^+_{\bar{z}\bar{z}}\int_0^\infty d\omega_q\left(a_-(\omega_q\hat{x})e^{-i\omega_q u} - a_+(\omega_q\hat{x})^\dagger e^{i\omega_q u}\right)$$

... so that it selects a certain low frequency mode...

$$\int du u \partial_u C_{\bar{z}\bar{z}} = \frac{i\kappa}{8\pi} \hat{\varepsilon}^{+}_{\bar{z}\bar{z}} \lim_{\omega \to 0} (1 + \omega \partial_\omega) [a_-(\omega \hat{x}) - a_+(\omega \hat{x})^\dagger]$$

... which has a universal expression when inserted in amplitudes.

$$\langle out | a_\pm(q) \mathcal{S} | in \rangle = (S^{(0)\pm} + S^{(1)\pm}) \langle out | \mathcal{S} | in \rangle + \mathcal{O}(\omega)$$

$$S^{(0)\pm} = \frac{\kappa}{2} \frac{(p_k \cdot \varepsilon^\pm)^2}{p_k \cdot q} \quad S^{(1)\pm} = -i \frac{\kappa}{2} \sum_k \frac{p_{k\mu} \varepsilon^{\pm\mu\nu} q^\lambda J_{k\lambda\nu}}{p_k \cdot q}$$

1406.3312

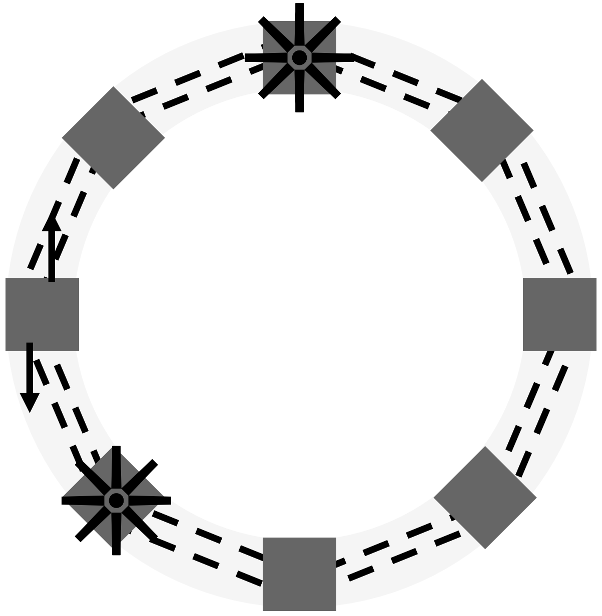
By matching the charge at $\mathcal{I}^+_-$ to $\mathcal{I}^-_+$ this turns into a Ward identity…

$$\langle \text{out}|\text{Q}^+(\text{Y})\mathcal{S} - \mathcal{S}\text{Q}^-(\text{Y})|\text{in}\rangle = 0$$

… and we have shown that superrotations are a physical symmetry.

1406.3312

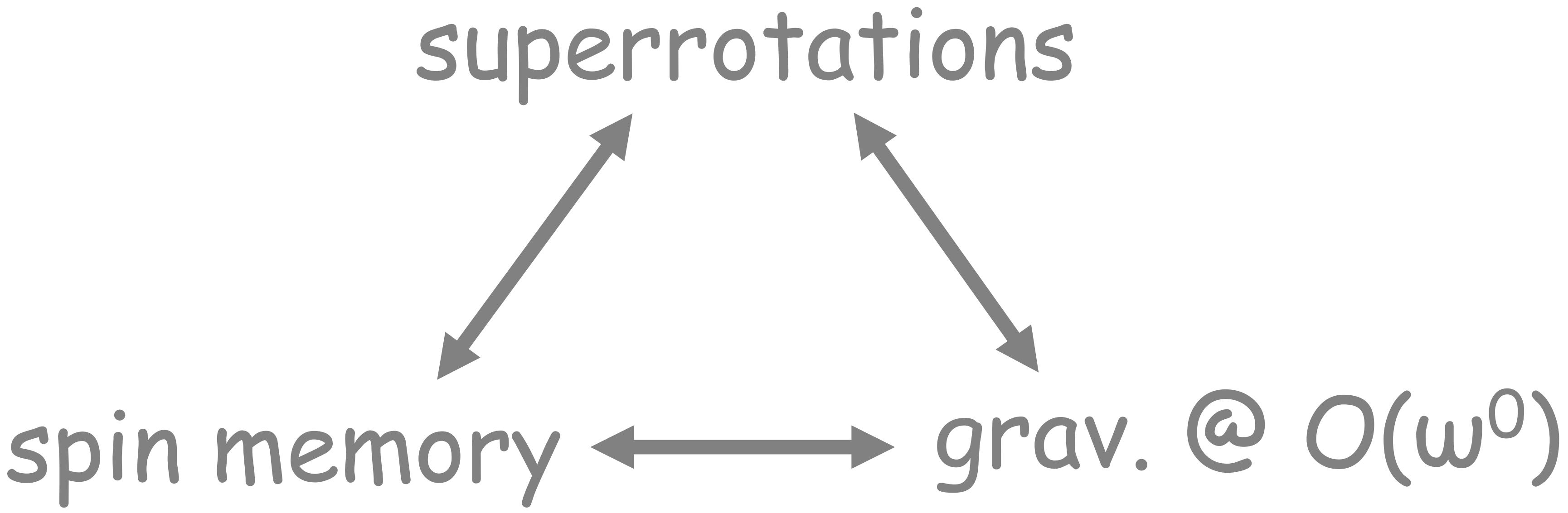
The classical observable that picks out this mode is a Sagnac effect.



It measures angular momentum flux and is called *spin memory*.

1502.06120, 1902.01840

We have completed a new iteration of the IR triangle…



… but more importantly this one appears to provide a key ingredient to a 2D dual.

1505.00716

For a particular choice of Y^z…

$$T_{zz} = 2iQ_S^+(Y^w = \frac{1}{z-w}, Y^{\bar{w}} = 0)$$

… the 4D superrotation Ward identity takes the form of a 2D stress tensor conformal Ward identity…

$$\langle T_{zz}\mathcal{O}_1 \ldots \mathcal{O}_n\rangle = \sum_k^n \left[\frac{h_k}{(z-z_k)^2} + \frac{1}{z-z_k}(\partial_{z_k} + h_k\Gamma^{z_k}_{z_k z_k} - |s_k|\Omega_{z_k})\right]\langle\mathcal{O}_1 \ldots \mathcal{O}_n\rangle$$

… *<u>but</u>* this assumes we can transform to a conformal basis where…

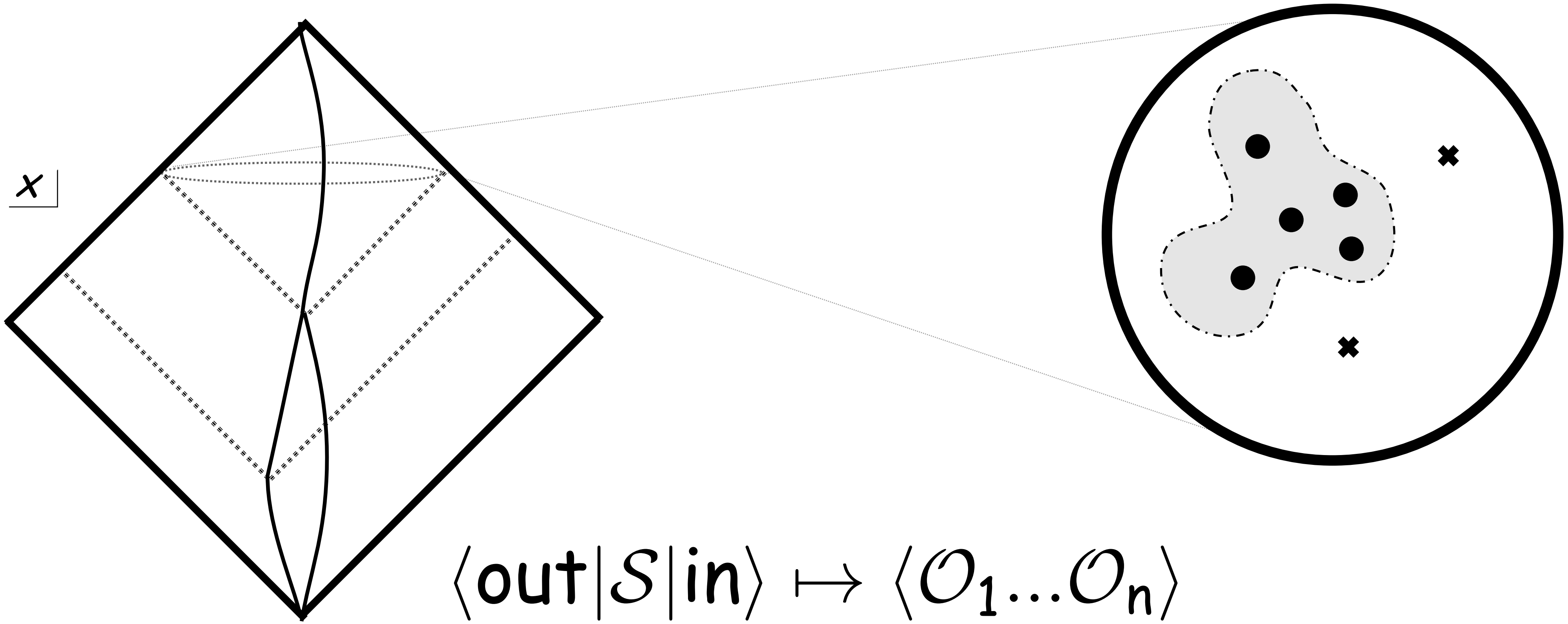
$$h_k \equiv \frac{1}{2}(s_k - \omega\partial_{\omega_k}), \quad \bar{h}_k \equiv \frac{1}{2}(-s_k - \omega\partial_{\omega_k})$$

We can, using a spectrum that lies on the principal series…

$$h_k + \bar{h}_k = \Delta_k \in 1 + i\mathbb{R}$$

1705.01027

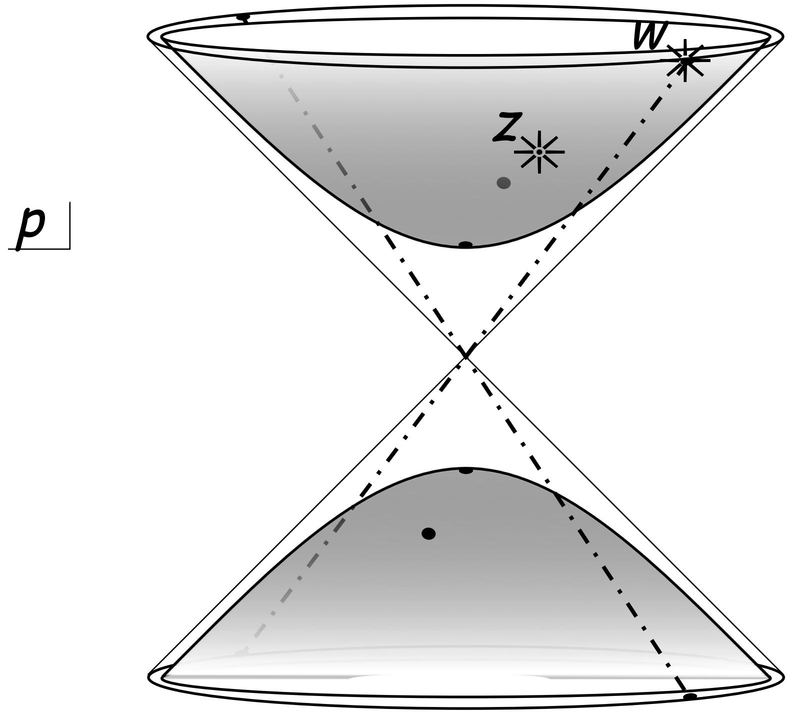
We want to convert our single particle states to local operators on the celestial sphere...



$$\langle \mathrm{out}|\mathcal{S}|\mathrm{in}\rangle \mapsto \langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle$$

1701.00049, 1706.03917

Recall the Lorentz group SO(3,1) is the Euclidean conformal group in 2D…



$$w \to w' = \frac{aw+b}{cw+d}$$

$$z \to z' = \frac{(az+b)(\bar{c}\bar{z}+\bar{d}) + a\bar{c}y^2}{|cz+d|^2 + |c|^2y^2}$$

$$\bar{z} \to \bar{z}' = \frac{(\bar{a}\bar{z}+\bar{b})(cz+d) + \bar{a}cy^2}{|cz+d|^2 + |c|^2y^2}$$

$$y \to y' = \frac{y}{|cz+d|^2 + |c|^2y^2}$$

SL(2,C)

1701.00049, 1706.03917

Demanding the Lorentz transformation properties of a solution to the KG eq…

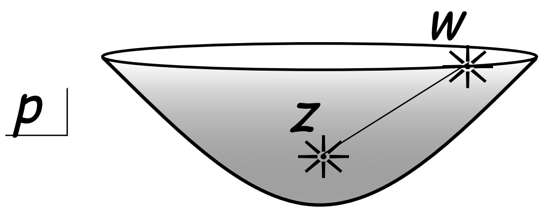
$$\varphi_{\Delta,m}\left(\Lambda^{\mu}{}_{\nu}X^{\nu};\frac{aw+b}{cw+d},\frac{\bar{a}\bar{w}+\bar{b}}{\bar{c}\bar{w}+\bar{d}}\right)=|cw+d|^{2\Delta}\varphi_{\Delta,m}\left(X^{\mu};w,\bar{w}\right)$$

And similarly for an on-shell Maxwell field…

$$V^{\Delta\pm}_{\mu J}\left(\Lambda^{\mu}{}_{\nu}X^{\nu};\frac{aw+b}{cw+d},\frac{\bar{a}\bar{w}+\bar{b}}{\bar{c}\bar{w}+\bar{d}}\right)=(cw+d)^{\Delta+J}(\bar{c}\bar{w}+\bar{d})^{\Delta-J}\,\Lambda_{\mu}{}^{\rho}V^{\Delta\pm}_{\rho J}(X^{\mu};w,\bar{w})$$

1701.00049, 1706.03917

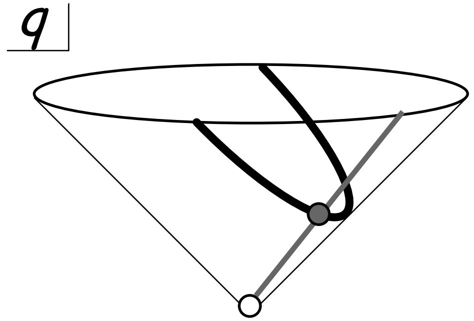
We can find the conformal primaries in terms of bulk-to-boundary propagators…



$$\varphi^{\pm}_{\Delta,m}(X^\mu; w, \bar{w}) = \int_0^\infty \frac{dy}{y^3} \int dz d\bar{z}\, G_\Delta(y, z, \bar{z}; w, \bar{w}) \exp\left[\pm i m\, \hat{p}^\mu(y, z, \bar{z})\, X_\mu\right]$$

… or, up to gauge equivalence, by a Mellin transform when m=0…

$$V^{\Delta,\pm}_{\mu J}(X^\mu; z, \bar{z}) = \frac{\partial_J q_\mu}{\sqrt{2}} \int_0^\infty d\omega\, \omega^{\Delta-1}\, e^{\pm i\omega q\cdot X - \varepsilon\omega}$$



1701.00049, 1706.03917

We can apply this transform to a free plane wave…

$$\varphi^{\pm}_{\Delta,m}(X^\mu; w, \bar{w}) = \frac{4\pi}{im} \frac{(\sqrt{-X^2})^{\Delta-1}}{(-X^\mu q_\mu \mp i\varepsilon)^\Delta} K_{\Delta-1}\left(im\sqrt{-X^2}\right)$$

… to get the position space wavefunction…

$$V^{\Delta,\pm}_{\mu J}(X^\mu; z, \bar{z}) = \mathcal{N} \frac{\partial_J q_\mu}{(-q \cdot X \mp i\varepsilon)^\Delta}$$

1701.00049, 1706.03917

… to a known amplitude …

$$\tilde{\mathcal{A}}_{\Delta_1,\cdots,\Delta_n}(w_i,\bar{w}_i) = \prod_{i=1}^{n}\int_0^\infty \frac{dy_i}{y_i^3}\int dz_i d\bar{z}_i\, G_{\Delta_i}(y_i,z_i,\bar{z}_i;w_i,\bar{w}_i)\,\mathcal{A}(m_j\hat{p}_j^\mu)$$

… to get the new amplitude in our conformal basis…

$$\tilde{\mathcal{A}}_{J_1\cdots J_n}(\lambda_j,z_j,\bar{z}_j) = \prod_{i=1}^{n}\int_0^\infty d\omega_i\,\omega_i^{\lambda_i}\,\mathcal{A}_{\ell_1\cdots\ell_n}(\omega_j,z_j,\bar{z}_j)$$

1701.00049, 1706.03917

… or to a single-particle state…

$$|\lambda, z, \bar{z}; h\rangle = \int_0^\infty d\omega\, \omega^{i\lambda} |\omega \hat{q}; h\rangle$$

… to give us insight into our construction.

From the position space wavefunctions we are able to…

- Check if their norm is finite
- Determine a complete basis

… and we note that we find the principal series spectrum and use tools from the embedding space formalism.

From the S-matrix transform we are able to see how familiar amplitudes behave in this basis…

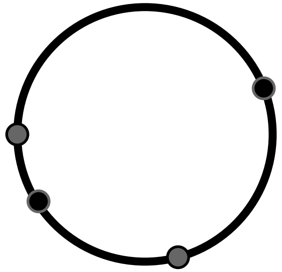
- Covariance of a conformal correlator of quasi-primaries is guaranteed…

- But 'atypical' singular structures in 2D appear due to 4D translation invariance.

1706.03917

For example the tree-level color-ordered 4pt MHV amplitude in the Mellin basis…

$$\tilde{\mathcal{A}}_{--++}(\lambda_i, z_i, \bar{z}_i) = -\frac{\pi}{4}\delta(\sum_k \lambda_k)\delta\left(\frac{|z-\bar{z}|}{2}\right)\left(\prod_{i<j}^{4} z_{ij}^{\frac{h}{3}-h_i-h_j}\bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j}\right) z^{\frac{5}{3}}(1-z)^{-\frac{1}{3}}$$
$$\times \prod_{i=1}^{4} 1_{[0,1]}(\sigma_{*i})$$



…has support on a celestial circle where the cross ratio, z, is real.

1706.03917

So we have constructed the 4D→2D map predicted by superrotations. Given what we have seen thus far some reasonable things to be (keep) doing include…

- Reinterpret currents as conformal primaries
- Evaluate more amplitudes
- Continue to AdS_3/CFT_2 via hyperbolic foliations

… and we will close with a few comments on some other potentially fruitful directions.

The 4D Hilbert space provides a unitary representation of the Poincaré group. In particular the Hermitian SO(3,1) generators…

$$M^{\mu\nu} = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & J_3 & -J_2 \\ -K_2 & -J_3 & 0 & J_1 \\ -K_3 & J_2 & -J_1 & 0 \end{pmatrix}$$

… can be combined into the non-Hermitian…

$$J_i^{\pm} = \frac{1}{2}(J_i \pm iK_i)$$

We can then form yet another combination…

$$L_0 = J_3^-, \quad L_{-1} = J_1^- + iJ_2^-, \qquad L_1 = J_1^- - iJ_2^-$$

$$\bar{L}_0 = -J_3^+, \quad \bar{L}_{-1} = J_1^+ - iJ_2^+, \qquad \bar{L}_1 = J_1^+ + iJ_2^+$$

… and using the behavior of the Lorentz generators under parity…

$$PJ^iP^{-1} = J^i, \qquad PK^iP^{-1} = -K^i$$

… we find that under parity + Hermitian conjugation…

$$\mathcal{O}^c = P\mathcal{O}^\dagger P^{-1}$$

$$L_0^c = L_0, \quad L_{-1}^c = L_1, \quad L_1^c = L_{-1}$$
$$\bar{L}_0^c = \bar{L}_0, \quad \bar{L}_{-1}^c = \bar{L}_1, \quad \bar{L}_1^c = \bar{L}_{-1}$$

… which would be like considering the out states (which seems related to antipodal matching) …

$$\langle\langle \text{out}| = (P|\text{in}\rangle)^\dagger$$

In position space we would use the explicit form of the vector fields…

$$\ell_0 = \frac{1}{2} y\partial_y + z\partial_z \qquad \bar{\ell}_0 = \frac{1}{2} y\partial_y + \bar{z}\partial_{\bar{z}}$$

$$\ell_{-1} = \partial_z \qquad \bar{\ell}_{-1} = \partial_{\bar{z}}$$

$$\ell_1 = y(z\partial_y - y\partial_{\bar{z}}) + z^2\partial_z \qquad \bar{\ell}_1 = y(\bar{z}\partial_y - y\partial_z) + \bar{z}^2\partial_{\bar{z}}$$

… to talk about the conformal covariance of our wavefunctions…

$$u = \frac{y^2 + |z|^2}{y} \qquad \ell_0 u^n = \bar{\ell}_0 u^n = \frac{n}{2} u^n$$

$$\ell_1 u^n = \bar{\ell}_1 u^n = 0$$

Now we can construct our Mellin states using a boost operator ...

$$\int d\beta e^{-i\lambda\beta} e^{i\beta\hat{p}\cdot K} |(1, \hat{p}), h\rangle$$

... and see that annihilation by L_1 and $\bar{L}_1$ comes from the ISO(2) representation...

$$L_1 = \bar{L}_1^\dagger = \frac{1}{2}((J_1 - K_2) - i(J_2 + K_1))$$

$$\hat{p} = (0, 0, 1)$$

Meanwhile a translation acts as…

$$e^{ix\cdot P}\int_{-\infty}^{\infty} d\beta e^{-i\lambda\beta} U(e^{i\beta\hat{p}\cdot K})|\hat{p},h\rangle = \int_{-\infty}^{\infty} d\beta e^{-i\lambda\beta + i(x\cdot\hat{p})e^{\beta}} U(e^{i\beta\hat{p}\cdot K})|\hat{p},h\rangle$$

… so we see the expected weight shift for infinitesimal translations.

Looking forward, we have some very interesting results thus far…

Looking forward, we have some very interesting results thus far…

... and they seem ready for a more formal attack on the structure of the 2D CFT.

Looking forward, we have some very interesting results thus far…

… and they seem ready for a more formal attack on the structure of the 2D CFT.

We would love to apply techniques from the CFT Bootstrap to say something general about our Mellin amplitudes…

Looking forward, we have some very interesting results thus far...

... and they seem ready for a more formal attack on the structure of the 2D CFT.

We would love to apply techniques from the CFT Bootstrap to say something general about our Mellin amplitudes...

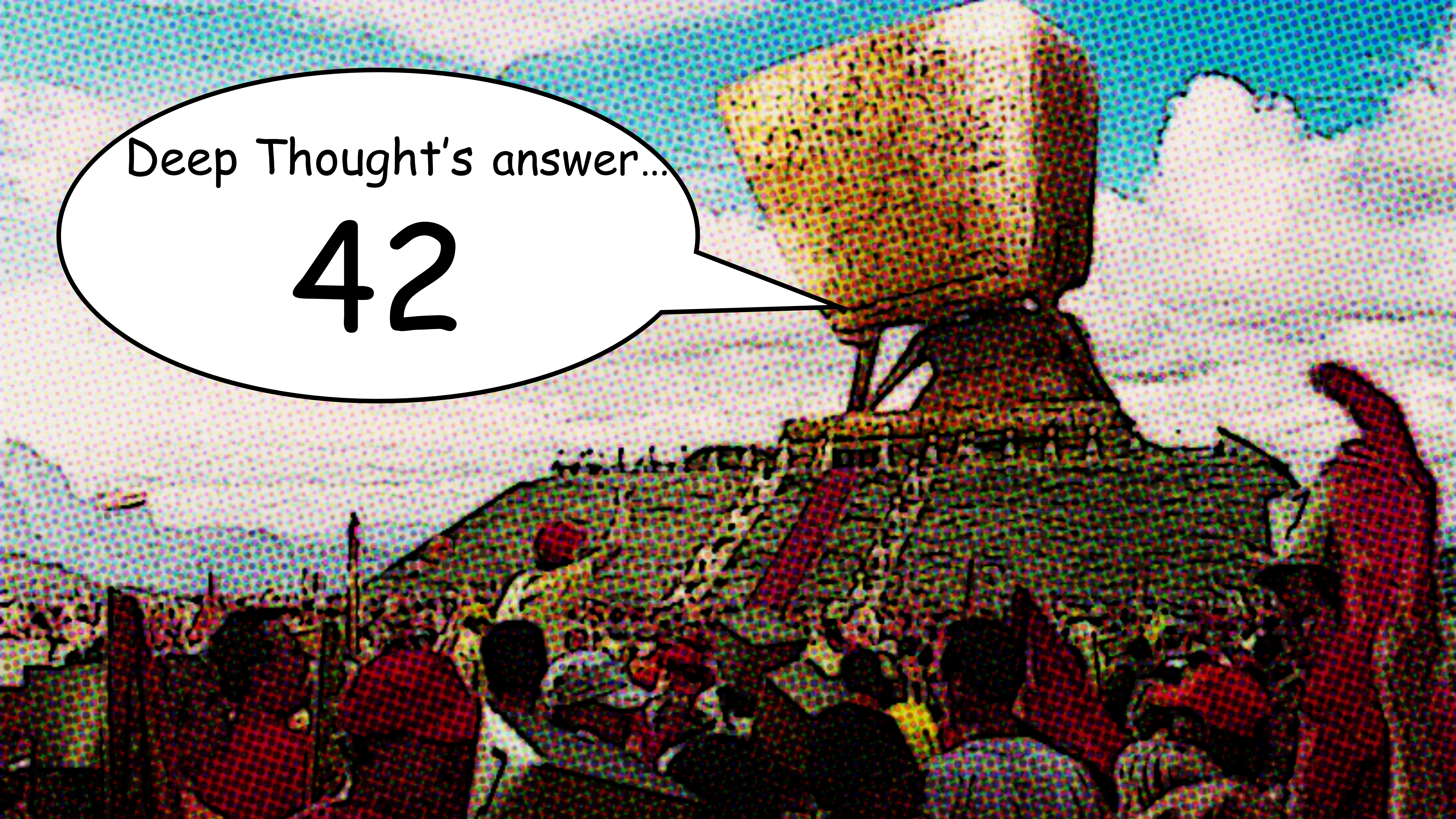
... *The Boo<u>st</u>strap* ???

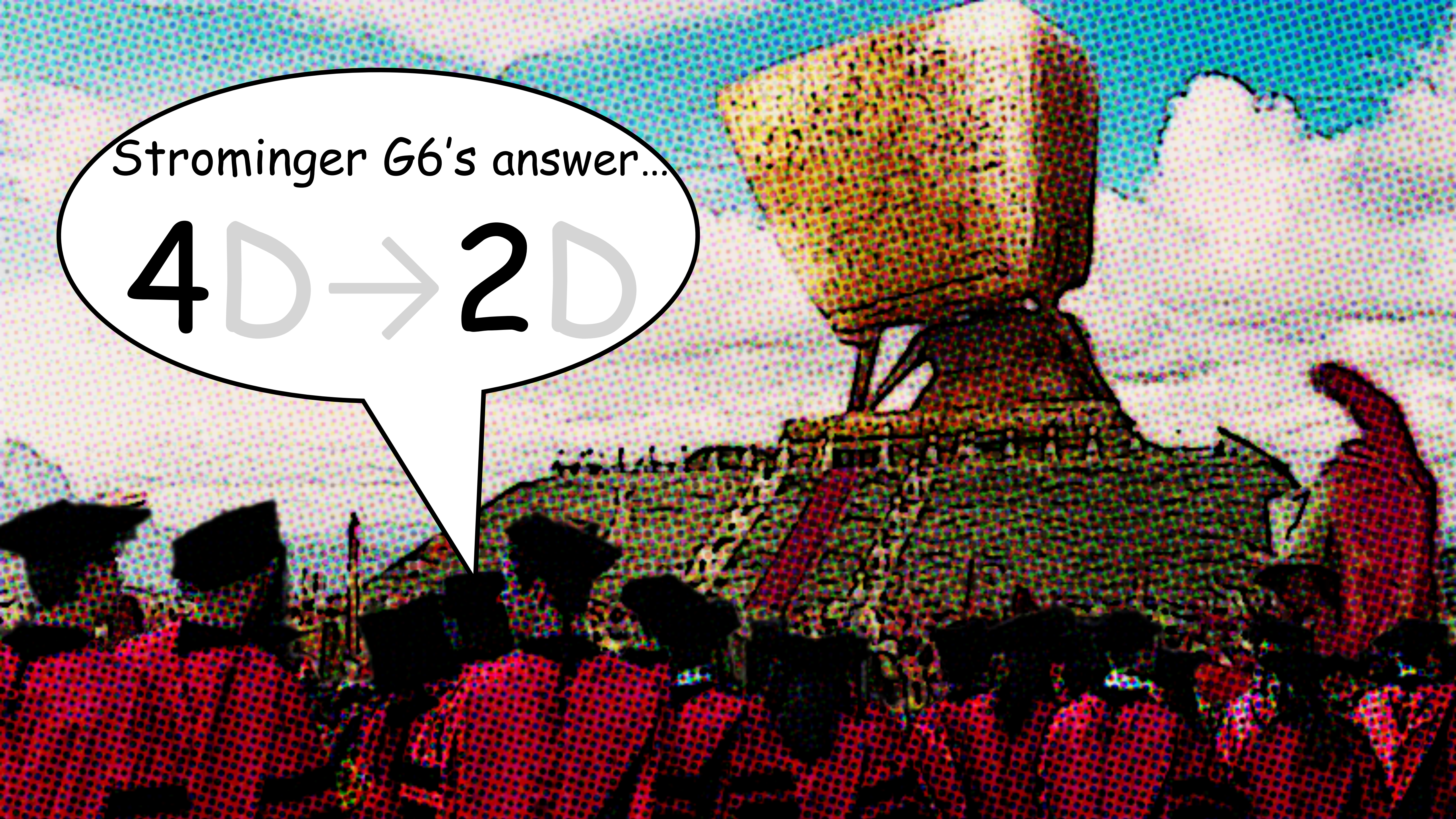
Thank you!

I hope Disney, BBC, Warner Bros., Square Enix & the other copyright holders appreciate this parody :oP

What a G1 hopes to find…

"the ultimate answer to life,
the universe and everything…"


Deep Thought's answer...
42


Strominger G6's answer...
4D→2D