



CENTER FOR THE FUNDAMENTAL LAWS OF NATURE



Harvard University
High Energy Theory Group

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Dear All,

Tomorrow Monday April 21 we will have our LAST TALK of the year.

The speaker will be Sabrina Pasterski and her title is "Gaussian Measures in Quantum Theory"

I hope to see many of you there. There will be food

Best,
Achilleas

Gaussian Measures in Quantum Theory

Arthur Jaffe 365a

↳ Reflection Postulate: going from Euclidean to Lorentzian
 Today: Euclidean free field Green's function \rightarrow field $\bar{\Phi}$ defined via a Gaussian measure $d\mu_c$
 continuity and interpretation of a "commutator"
 set $\Phi(t; x_i) \Rightarrow \sum_{i=1}^n \delta(\bar{\Phi}(t_i) - x_i) d\mu_c$

Interpret this probability two ways: \Rightarrow 0+1 dimensional free field theory
 (can add interactions, get LSZ)

\Rightarrow Evolution of S.H.O.

Goal: Show how $d\mu_c$ connects 0+1 dimensional free field theory to QM simple harmonic oscillator

Fields with a Gaussian Measure

consider: $S(\xi) = \int e^{i\xi \cdot x} \nu(x) dx$

Bochner's theorem $\Rightarrow S(\xi)$ a positive, continuous function on \mathbb{R}^d normalized to $S(0) = 1$
 is the Fourier transform of a positive measure $d\nu(x) = \nu(x) dx$ (Borel measure)
 ν probability distribution on \mathbb{R}^d

generalization: $f \in L^2(\mathbb{R}^d)$ in space of test functions \mathcal{H}
 paired to $\bar{\Phi} \in \mathcal{H}'$ continuous dual of distributions

$$S(\bar{\Phi}f) = \int e^{i\bar{\Phi}(\xi)\xi} d\mu_c(\bar{\Phi}) = e^{-\frac{\lambda^2}{2} \langle \bar{\Phi}, f \rangle_2},$$

$\underbrace{\text{defines } d\mu_c(\bar{\Phi}) \text{ completely}}$
 $\text{in terms of covariance } C$

$$\begin{aligned} \bar{\Phi}(\xi) &= \langle f, \bar{\Phi} \rangle = \int \bar{\Phi}(x) f(x) dx \\ &\text{in 0+1 dimensions for } f(t) = \delta(t-t_0) \\ \bar{\Phi}(\xi) &= \bar{\Phi}(t_0) \end{aligned}$$

$$\Rightarrow -\frac{d^2}{d\xi^2} S(\bar{\Phi}f) \Big|_{\xi=0} = \int \bar{\Phi}(f)^2 d\mu_c(\bar{\Phi})$$

$$= \langle \bar{\Phi}, f \rangle_{L^2}$$

$$C = (-\Delta + m^2)^{-1} \quad \Delta = \sum_{i=1}^3 \partial_i^2 \quad \text{Yukawa potential in } \mathbb{R}^3$$

$$\Rightarrow (-\frac{\partial^2}{\partial t^2} + m^2)^{-1} \text{ in 0+1 dimensions trait}$$

define $C(t-t')$ as the kernel of C

$$(Cf)(t) = \int C(t-t') f(t') dt'$$

$$\text{now } \int \bar{\Phi}(\xi)^2 d\mu_c(\bar{\Phi}) = \langle \bar{\Phi}, f \rangle_2,$$

$$= \int dt' dt \bar{\Phi}(t) \bar{\Phi}(t') C(t-t')$$

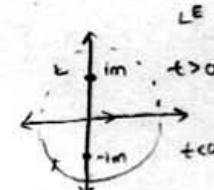
$$f = S(t-t_0) \Rightarrow \int \bar{\Phi}(t)^2 d\mu_c(\bar{\Phi}) = C(0)$$

$$f = \alpha \delta(t-t_0) + \beta \delta(t-t_1) \Rightarrow \bar{\Phi}(t) = \alpha \bar{\Phi}(t_0) + \beta \bar{\Phi}(t_1)$$

$$\langle \bar{\Phi}, f \rangle_2 = (\alpha^2 + \beta^2) C(0) + 2\alpha\beta C(t_0 - t_1)$$

$$\Rightarrow \int \bar{\Phi}(t_0) \bar{\Phi}(t_1) d\mu_c(\bar{\Phi}) = C(t_0 - t_1)$$

$$\begin{aligned} C &= \frac{1}{2\pi} \int \frac{e^{iEt}}{E^2 + m^2} dE \\ &\approx (E+i\epsilon)(E-i\epsilon) \\ &= \begin{cases} \frac{2\pi i}{2\pi} \frac{e^{iEim}}{2im} = \frac{e^{-m\epsilon}}{2m} & t > 0 \\ \frac{-2\pi i}{2\pi} \frac{e^{iEim}}{-2im} = \frac{e^{m\epsilon}}{2m} & t < 0 \end{cases} \\ &\Rightarrow \frac{e^{-m|\epsilon|}}{2m} \quad \epsilon \in \mathbb{R} \end{aligned}$$



Continuity of Φ

consider the expectation value $\int \bar{\Phi}(t^+) \bar{\Phi}_t \bar{\Phi}(t^-) - \bar{\Phi}_t \bar{\Phi}(t^+) \bar{\Phi}(t^-) d\mu_c(\bar{\Phi})$

where t^+ , t^- refer to $t \rightarrow t \pm \Delta t$ for $\Delta t \rightarrow 0^+$

$$\Rightarrow \lim_{\Delta t \rightarrow 0^+} \int \bar{\Phi}(t+\Delta t) \left\{ \frac{\bar{\Phi}(t+\Delta t) - \bar{\Phi}(t)}{\Delta t} \right\} - \left\{ \frac{\bar{\Phi}(t+\Delta t) - \bar{\Phi}(t)}{\Delta t} \right\} \bar{\Phi}(t) d\mu_c(\bar{\Phi})$$

$$= \lim_{\Delta t \rightarrow 0^+} \int \frac{(\bar{\Phi}(t+\Delta t) - \bar{\Phi}(t))^2}{\Delta t} d\mu_c(\bar{\Phi})$$

$$= 2 \lim_{\Delta t \rightarrow 0^+} \frac{C(\Delta t)}{\Delta t}$$

$$= \frac{i}{m} \lim_{\Delta t \rightarrow 0^+} \frac{1 - e^{-m\Delta t}}{\Delta t}$$

= i Hölder continuous with exponent $1/2$

non-zero "commutation" \leftrightarrow Hölder continuous, not differentiable
 independent of m (would generalize to $\mu = \sqrt{E^2 + m^2}$ in \mathbb{R}^d , $\partial_\mu \rightarrow \square$)

\square_t since we have Wick rotated
 $(\tau = it \rightarrow \frac{2\pi}{\lambda} \tau = i\frac{2\pi}{\lambda} t)$
 $\delta(\tau - \tau') = -i S(t-t')$

As a Green's function

$$(\square + m^2) C(t-t') = \delta(t-t')$$

can interpret this as a two-point function

$$(\square + m^2) \int \bar{\Phi}(t) \bar{\Phi}(t') d\mu_c(\bar{\Phi}) = \delta(t-t')$$

comes from correlations rather than time ordering

Computing $\int \prod_{i=1}^n \delta(\bar{x}(t_i) - x_i) d\mu_c(\bar{x}) \equiv \rho_t(x)$

$$\begin{aligned} & \Rightarrow \frac{1}{(2\pi)^n} \int d\mu_c d\bar{k} e^{\sum_{j=1}^n i k_j (\bar{x}(t_j) - x_j)} \\ & = \frac{1}{(2\pi)^n} \int d\bar{k} e^{-i \bar{k} \cdot \bar{x}} \underbrace{\int d\mu_c e^{\sum_{j=1}^n i k_j \bar{x}(t_j)}}_{f = \sum_{j=1}^n k_j \delta(t - t_j)} \\ & = \frac{1}{(2\pi)^n} \int d\bar{k} e^{-i \bar{k} \cdot \bar{x}} e^{-\frac{1}{2} \bar{k}^T C \bar{k}} \\ & = \frac{1}{\sqrt{(2\pi)^n \det C}} e^{-\frac{1}{2} \bar{x}^T C^{-1} \bar{x}} \end{aligned}$$

$\Rightarrow e^{-\frac{1}{2} \langle \bar{f}, \bar{f} \rangle}$ from $S(f)$ definition

$$\begin{aligned} & = e^{-\frac{1}{2} k_i C_{ij} k_j} \text{ where } C_{ij} = C(t_i - t_j) \\ & \text{from } \int dt' dt f(t) f(t') C(t - t') \\ & = \frac{e^{-m \frac{1}{2} |t_i - t_j|}}{2\pi} \end{aligned}$$

Interpreting $\rho_t(x)$

- $\rho_t(x) \geq 0$ and $\int_{\mathbb{R}^n} \rho_t(x) dx = 1$

think of $\prod_{i=1}^n \delta(\bar{x}(t_i) - x_i)$ as specifying each component of \bar{x}

$\bar{x}(t)$ as a path $\Rightarrow \rho_t(x)$ probability densities for path to pass through x at times t

- Large N /continuum limit $\rho \Rightarrow 0+1$ dimensional QFT, get Wick rotated LSZ

- Explicit calculation for finite $N \Rightarrow \rho$ related to evolution of the quantum simple harmonic oscillator

Field Theory Interpretation

$c(t) = (-\Delta + m^2)^{-1} \delta(t)$ is the impulse response to $[-\Delta + m^2]^{-1}$

$$\bar{x}^T C^{-1} \bar{x} \rightarrow \int dt \{x(t) [-\Delta + m^2] x(t)\}$$

gives $\rho = \int \delta(\bar{x}(t) - x(t)) d\mu_c \propto \sqrt{\det[-\Delta + m^2]} e^{-\frac{1}{2} \int dt \{x(t) [-\Delta + m^2] x(t)\}}$ (can set proportionality constant to 1 using Gaussian integral and convention $\int dx d\mu_c \delta(\bar{x}(t) - x(t)) = \int d\mu_c$)

[rescaling x
make dimensionless]

$$S_0 = \frac{1}{2} \dot{x}^2 + \frac{1}{2} m^2 x^2$$

$$L = -\dot{x}(x \rightarrow i\tau)$$

$$S_E [x(\tau)] = \int d\tau \left\{ \frac{1}{2m} \dot{x}^2 + V(x) \right\}$$

$$IS = \frac{i}{2} \int dt \phi(-\dot{x}^2 + m^2) \phi$$

$$IS = \frac{1}{2} \int d\tau \phi(-\dot{x}_\tau^2 + m^2) \phi$$

$$= \frac{1}{2} \int d\tau \phi(-\dot{x}_\tau^2 + m^2) \phi$$

$$S_c \rightarrow S_0 + \int dt L_{int}[x(t)]$$

weight expectation values

$$\text{with a modified measure } d\tilde{\mu}_c = d\mu_c e^{-\int dt_0 L_{int}[\phi(t_0)]}$$

$$\langle x(t) x(t') \rangle \equiv \frac{\iint dx \{x(t) x(t')\} d\tilde{\mu}_c \delta(\bar{x} - x)}{\iint dx d\tilde{\mu}_c \delta(\bar{x} - x)}$$

$$= \frac{\int \bar{x}(t) \bar{x}(t') e^{-\int dt_0 L_{int}[\phi(t_0)]} d\mu_c}{\int e^{-\int dt_0 L_{int}[\phi(t_0)]} d\mu_c}$$

from λ derivatives of $S(\lambda f) = \int e^{i\bar{x}(t)f} d\mu_c(\bar{x}) = e^{-\frac{\lambda^2}{2} \langle \bar{f}, \bar{f} \rangle}$, $\lambda \rightarrow 0$

$$\Rightarrow \int \bar{x}(t)^n d\mu_c(\bar{x}) = \frac{(2n)!}{2^n n!} \langle \bar{f}, \bar{f} \rangle^n$$

$$\int \bar{x}(t)^{n+1} d\mu_c(\bar{x}) = 0$$

$$\int \sum_{i=1}^n \bar{x}(v_i) d\mu_c(\bar{x}) = \sum_{\text{pairings}} \langle \bar{x}(v_{i,1}) \bar{x}(v_{i,2}) \rangle_c \dots \langle \bar{x}(v_{i,n-1}) \bar{x}(v_{i,n}) \rangle_c$$

$$\langle \bar{x}(t) \bar{x}(t') e^{-\int dt_0 L_{int}[\phi(t_0)]} \rangle_c = \langle \bar{x}(t) \bar{x}(t') \rangle_c \langle e^{-\int dt_0 L_{int}[\phi(t_0)]} \rangle_c + \dots$$

$\bar{x}(t)$ contracted with t , from $e^{-\int dt_0 L_{int}} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int dt_0 \dots dt_m L_{int}[\phi(t_1)] \dots L_{int}[\phi(t_m)]$

since $(-\Delta + m^2)_t \langle \bar{x}(t) \bar{x}(t') \rangle_c = \delta(t - t')$ the first term gives a contact term, while the others give $-\frac{\delta I_{mt}}{8\Phi(c)}$

$$\Rightarrow (-\Delta + m^2)_t \langle x(t) x(t') \rangle = \delta(t - t) - \langle \frac{\delta I_{mt}}{8\Phi(c)} x(t') \rangle$$

$$L \rightarrow \frac{L}{n} (-\Delta + m^2) C(t-t) \rightarrow h \delta(t-t)$$

contact term $\propto h$, interaction term has $\frac{h}{n} = 1$

(normalization of x : want $[x] = [t] = [m]^{-1}$ to think of x field as the position
 $x \rightarrow \frac{x}{m} \Rightarrow S_E = \int dt \frac{1}{2m} \dot{x}^2 + \frac{1}{2} m x^2$ correction to SHO with $w=1$.)

Simple Harmonic Oscillator Interpretation

Consider $H_0 = \frac{1}{2}(-\frac{d^2}{dx^2} + m^2 x^2 - m)$ $\leftrightarrow m \cdot (\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - \frac{1}{2})$, $w=1 \Rightarrow$ spectrum $n \in \mathbb{Z}^+$ of Hamiltonian gives $m\mathbb{Z}^+$ spectrum for H_0

(Jaffe interprets as scaling $x \propto \sqrt{m}$ so $m=1$ and now having $\omega=m$)

$$\mathcal{J}_0(x) = \left(\frac{m}{\pi}\right)^{1/4} e^{-mx^2/2}$$

$$(e^{-tH_0} f)(x) = \int_{-\infty}^{\infty} B_t(x, x') f(x') dx' \quad \text{Boltzmann integral kernel (Mehler)} \quad t > 0 \quad \text{heat kernel}$$

$$B_t(x, x') = \left(\frac{me^{mt}}{2\pi \sinh(mt)}\right)^{1/2} e^{-\frac{m(x^2+x'^2)\cosh(mt)-2mx x'}{2\sinh(mt)}}$$

$$\rho_t(x) = \int_{i=1}^N \delta(\bar{x}(t_i) - x_i) d\mu_c(\bar{x}) = \frac{1}{\sqrt{(m\pi)^N}} e^{-\frac{1}{2}\bar{x}^T C^T \bar{x}} \quad C_{ij} = \frac{e^{-m(t_i-t_j)}}{2m}$$

$$N=1 \Rightarrow \rho_t(x) = \sqrt{\frac{m}{\pi}} e^{-mx^2} = \mathcal{J}_0(x)^2$$

for $N > 1$ it is convenient to define $C = 2mC$

$$\rho_t(x) = \left(\frac{m}{\pi}\right)^{N/2} \frac{e^{-m\bar{x}^T C \bar{x}}}{\sqrt{N!}}$$

$$N=2 \Rightarrow \rho_{t_1, t_2}(x_1, x_2) = \frac{m}{\pi} \frac{e^{-m(x_1^2 + x_2^2) C_2^{-1}(x_1, x_2)}}{\sqrt{1 - e^{-2m(t_2-t_1)}}}$$

$$C = \begin{pmatrix} 1 & e^{-m(t_2-t_1)} & e^{-m(t_3-t_1)} & \dots \\ e^{-m(t_2-t_1)} & 1 & e^{-m(t_3-t_2)} & \dots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \quad t_1 < t_2 < t_3 < \dots < t_N$$

$$C_2^{-1} = \begin{pmatrix} 1 & e^{-m(t_2-t_1)} \\ e^{-m(t_2-t_1)} & 1 \end{pmatrix}$$

$$C_2^{-1} = \frac{1}{1 - e^{-2m(t_2-t_1)}} \begin{pmatrix} 1 & -e^{-m(t_2-t_1)} \\ -e^{-m(t_2-t_1)} & 1 \end{pmatrix}$$

$$= \sqrt{\frac{m}{\pi}} \left(\frac{me^{-m(t_2-t_1)}}{2\pi \sinh(m(t_2-t_1))} \right)^{1/2} e^{-\frac{m(x_1^2 + x_2^2 - 2x_1 x_2 e^{-m(t_2-t_1)})}{(1 - e^{-2m(t_2-t_1)})}}$$

$$\sqrt{\frac{m}{\pi}} e^{\frac{m(x_1^2 + x_2^2)}{2}} \rho_{t_1, t_2}(x_1, x_2) = B_{t_2-t_1}(x_1, x_2)$$

$$\Rightarrow \rho_{t_1, t_2}(x_1, x_2) = \mathcal{J}_0(x_1) B_{t_2-t_1}(x_1, x_2) \mathcal{J}_0(x_2)$$

General N use induction

$$C_N = \begin{pmatrix} C_{N-1} & \delta \\ \delta^T & 1 \end{pmatrix} \Rightarrow C_N^{-1} = \begin{pmatrix} C_{N-1}^{-1} + \frac{(C_{N-1}^{-1} \delta)(C_{N-1}^{-1} \delta)^T}{\mu} & -\frac{(C_{N-1}^{-1} \delta)}{\mu} \\ -\frac{(C_{N-1}^{-1} \delta)}{\mu} & \frac{1}{\mu} \end{pmatrix} \quad \mu = 1 - \delta^T (C_{N-1}^{-1} \delta)$$

$$\delta = \begin{pmatrix} e^{-m(t_N-t_1)} \\ \vdots \\ e^{-m(t_N-t_{N-1})} \end{pmatrix}$$

$$C_{N-1}^{-1} \delta = \chi \Rightarrow \delta = C_{N-1} \chi = C_{N-1} \begin{pmatrix} \chi \\ \vdots \\ e^{-m(t_N-t_{N-1})} \end{pmatrix}$$

since last column in C_{N-1} is $\begin{pmatrix} e^{-m(t_{N-1}-t_1)} \\ \vdots \\ 1 \end{pmatrix}$ and C_{N-1} is invertible

$$(C_{N-1}^{-1} \delta)_j = e^{-m(t_N-t_{N-j})} \delta_{j, N-1} \text{ picks out one component}$$

$$\Rightarrow \mu = 1 - e^{-2m(t_N-t_{N-1})} \\ (x_N^T C_N^{-1} x_N) = x_N^T C_{N-1}^{-1} x_{N-1} + \frac{1}{\mu} (e^{-2m(t_N-t_{N-1})} x_{N-1}^2 - 2e^{-m(t_N-t_{N-1})} x_{N-1} x_N + x_N^2)$$

$$= x_N^T C_{N-1}^{-1} x_{N-1} + \frac{1}{\mu} (e^{-2m(t_N-t_{N-1})} x_{N-1}^2 - 2e^{-m(t_N-t_{N-1})} x_{N-1} x_N + x_N^2)$$

$$\rho_N = \left(\frac{m}{\pi}\right)^{1/2} \cdot \left(\frac{m}{\pi}\right)^{N-1} \frac{1}{\sqrt{N! \det C_{N-1}}} \frac{1}{\sqrt{\mu}} e^{-\frac{m(x_N^2 + e^{-2m(t_N-t_{N-1})} x_{N-1}^2 - 2x_N x_{N-1} e^{-m(t_N-t_{N-1})})}{\mu}}$$

$$= \rho_{N-1} \times \left(\frac{m e^{-m(t_N-t_{N-1})}}{2\pi \sinh(m(t_N-t_{N-1}))}\right)^{1/2} e^{-\frac{m(x_N^2 + e^{-2m(t_N-t_{N-1})} x_{N-1}^2 - 2x_N x_{N-1} e^{-m(t_N-t_{N-1})})}{\mu}}$$

$$= \rho_{N-1} \times \sqrt{\frac{m}{\pi}} \rho_{t_N-t_{N-1}}(x_N, x_{N-1}) e^{-\frac{m x_{N-1}^2}{\pi}} \left(\frac{m}{\pi}\right)^{1/2} \mathcal{J}_0(x_{N-1})^{-2}$$

$$= \rho_{N-1} \times \mathcal{J}_0(x_{N-1})^{-1} B_{t_N-t_{N-1}}(x_{N-1}, x_N) \mathcal{J}_0(x_N)$$

$$\Rightarrow \boxed{\rho_N = \mathcal{J}_0(x_1) B_{t_2-t_1}(x_1, x_2) B_{t_3-t_2}(x_2, x_3) \dots B_{t_N-t_{N-1}}(x_{N-1}, x_N) \mathcal{J}_0(x_N)}$$

$t_1 < \dots < t_N$ $N-1$ factors of Mehler kernel

Technical Definitions

Reflection Positivity

1. vector space E with Hermitian form $\langle \cdot, \cdot \rangle$

2. reflection operator $\theta: \theta^2 = \text{Id}$ $\langle \theta f, g \rangle = \langle f, \theta g \rangle$

3. linear subspace $E_+ \subset E$ s.t. for all $f \in E_+$ $\langle f, \theta f \rangle \geq 0$

uses: start with E construct hilbert space H by taking E_+ and modding out N nullspace of $\langle \cdot, \theta \cdot \rangle$

Going from Euclidean to Lorentzian define θ by starting with a static manifold:

$$ds^2 = F(x) dt^2 + \sum_i G_{ii}(x) dx^i dx^i \quad M = \mathbb{R} \times \Sigma \quad \text{spatial hypersurface}$$

$\theta \in \text{isometries of } M$ fixes Σ (defined @ $t=0$) & exchanges $\mathcal{L}_\pm \Rightarrow M = \mathcal{L}_- \cup \Sigma \cup \mathcal{L}_+$

maps points to those with opposite time coordinate, have ∂_t as a killing vector field

construct a unitary of reflection positive theory, then analytically continue

why: conditions to go from a Euclidean theory to a Lorentzian one $F(x) \rightarrow -F(x)$

Gaussian Measure

1. covariance operator C on \mathcal{H} (test functions)

2. unique Gaussian measure $d\mu_C$ on dual \mathcal{H}' with C as its covariance operator

uniquely defined by covariance, mean 0

for C_v restriction to n -dim subspace $V \subset \mathcal{H}$ $dx_{C_v} = \frac{1}{\sqrt{(2\pi)^n \det C_v}} e^{-\frac{1}{2} \langle x, C_v^{-1} x \rangle} dx$

dx Lebesgue measure on V

dx_{C_v} Lebesgue measure on V' gaussian measure on \mathcal{H}'/V^\perp where $V^\perp = \{\theta \in \mathcal{H}: \langle \theta, v \rangle = 0\}$

$\gamma(A) = \frac{1}{\sqrt{2\pi}^n} \int_A e^{-\frac{1}{2} \|x\|^2} d\lambda(x) \quad A \in \mathcal{B}_\sigma(V)$ completion of Borel σ -algebra on V finite dimensional case

infinite dimensional case uses Borel cylinder sets $S_A; f_1, \dots, f_n \equiv \{\theta \in \mathcal{H}: (f_1(\theta), \dots, f_n(\theta)) \in A\}$

Borel set: set in a topological space formed from open sets via countable union, intersection or relative complement. Given a space X , σ -algebra from collection of all Borel sets

Resources

Glimm & Jaffe Quantum Physics

Jaffe's notes & those of students (Anderson)

Taylor UNC Ch 16 Wiener Measure & Brownian Motion

J.C. Zambrini multiple articles ex. Feynman Integrals, Diffusion Processes & Quantum Symplectic Two-forms
ex. of discrete version of commutator citing Feynman

Osterwalder & Schrader Axioms for Euclidean Green's Functions