Position as a Single-Valued Function of Time

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I present a geometrical illustration of how the speed limit in Special Relativity requires position to be a single-valued function of time in any reference frame.

I. BACKGROUND

In my paper “Motivating Special Relativity using Linear Algebra,” I derived the Lorentz transformations assuming that the speed of light is the same in any reference frame and that transformations of space-time coordinates are linear. In the \((\eta, \xi)\) basis where:

\[
\eta = \frac{ct - x}{\sqrt{2}}, \quad \xi = \frac{ct + x}{\sqrt{2}}
\]

(1)

a boost in velocity is described by:

\[
\begin{pmatrix}
\eta' \\
\xi'
\end{pmatrix} = \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix} \begin{pmatrix}
\eta \\
\xi
\end{pmatrix}
\]

\[
\lambda_1 = \sqrt{1 + \beta}, \quad \lambda_2 = \sqrt{1 - \beta}.
\]

(2)

The fact that \(|\beta| > 1\) makes \(\lambda_1\) and \(\lambda_2\) imaginary and justifies the notion of a speed limit in Special Relativity: \(|v| \leq c\). In this paper, I consider the implications of having the strict inequality \(|v| < c\) hold.

![Slope restrictions lead to properties of x(ct).](image)

The restriction that \(|\beta| < 1\) also implies that the trajectory of a particle in \((\eta, \xi)\) is strictly increasing as a function of \(\eta\). If its slope were zero at any point, the particle would be traveling at \(c\) in the \(-\hat{x}\) direction. If its slope were infinite, the particle would be traveling at \(c\) in the \(+\hat{x}\) direction. At any instant, the slope must be between these two values.

I will show that this implies that \(x = x(ct)\) for any reference frame by contradiction (the red curve in Figure 1b). If there exists a reference frame in which \(x'\) takes on more than one value for a given \(ct'\) then for a trajectory that is continuous (the particle does not jump between points in space and time) there will be some position along the particle’s path in the \((\eta, \xi)\) plane that has a tangent parallel to the average slope, which is the slope of the \(x'\) axis. This comes from an application of the mean value theorem of calculus.

I previously showed that the slope of any \(x'\) axis is negative, so this means that the slope of the trajectory in \((\eta, \xi)\) would also be negative, which is not allowed by \(|\beta| < 1\). This contradiction tells us that \(x\) is single-valued as a function of time in any reference frame. Special Relativity thus implies that the particle cannot occupy two positions at the same time.