Motivating Special Relativity using Linear Algebra

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I present a geometrical derivation of results from Special Relativity for a single spatial coordinate.

I. POSTULATES

In this paper, I derive results from Special Relativity using symmetry and the following postulates:

1. The speed of light is constant in any reference frame.

2. The transformation of space-time coordinates when changing between reference frames is linear.

\[ \begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} 1 & \beta / \sqrt{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix} \]

for some eigenvalues \( \lambda_1(v) \) and \( \lambda_2(v) \) which determine the scaling of the axes during a boost by \( v \).

In this \((\eta, \xi)\) basis, the slope of a particle traveling at a constant velocity is \( m = \frac{1 + \beta}{1 - \beta} \). A boost into a frame in which this particle has zero velocity must take this slope to 1:

\[ \frac{\Delta \xi'}{\Delta \eta'} = \frac{\lambda_2 \Delta \xi}{\lambda_1 \Delta \eta} = \frac{\lambda_2 \Delta x}{\lambda_1 \Delta t} = 1 \rightarrow \frac{\lambda_2}{\lambda_1} = 1 - \frac{\beta}{1 + \beta}. \] (2)

Now consider a switch in the sign of \( v \) by adding a second particle moving in the opposite direction. If both the \(+v\) and \(-v\) particles pass through \( x = 0 \) at \( t = 0 \), their positions at any time will be reflections of one another across the \( m = 1 \) diagonal in the \((\eta, \xi)\) plane, which corresponds to the time axis. If scaling \( \eta \) by \( \lambda_1(v) \) and \( \xi \) by \( \lambda_2(v) \) brings the \(+v\) particle’s space-time coordinate at a given \( ct \) to a particular \( ct' \) on the \( m = 1 \) diagonal during a \(+v\) boost, then scaling \( \eta \) by \( \lambda_2(v) \) and \( \xi \) by \( \lambda_1(v) \) will bring the corresponding space-time coordinate of a \(-v\) particle’s path to the same point on the \( m = 1 \) diagonal. Since this is equivalent to performing a \(-v\) boost instead, \( \lambda_1(-v) = \lambda_2(v) \).

Because \( \lambda_1(v) \) determines the scaling of the \( \eta \) axis during a boost by \( v \), a subsequent boost by \(-v\) should undo this rescaling, giving \( \lambda_1(-v) = 1/\lambda_1(v) \). This says that \( \lambda_1(v)\lambda_2(v) = 1 \): area elements are invariant under a boost. Solving for \( \lambda_1 \) and \( \lambda_2 \) gives:

\[ \lambda_1 = \sqrt{\frac{1 + \beta}{1 - \beta}} \quad \lambda_2 = \sqrt{\frac{1 - \beta}{1 + \beta}} \] (3)

where the limit of \( \lambda \rightarrow 1 \) for \( v \rightarrow 0 \) sets the overall sign of the eigenvalues.

From the definition of \( \eta \) and \( \xi \):

\[ \eta = \frac{ct - x}{\sqrt{2}} \quad \xi = \frac{ct + x}{\sqrt{2}} \] (4)

this transformation reduces to:

\[ ct' = \gamma(ct - \beta x) \quad x' = \gamma(x - \beta ct) \] (5)

in the \((ct, x)\) basis, with \( \gamma = \frac{1}{\sqrt{1 - \beta^2}} \). This completes a derivation of the standard Lorentz transformation for a single spatial coordinate in Special Relativity.

II. TRANSFORMING COORDINATES

When describing the motion of a particle along the \( x \) axis, it is convenient to plot position as a function of time. Figure 1a shows such a plot. The green line has slope \( \beta = \frac{v}{c} \) and describes the motion of a particle which moves at a constant velocity \( v \) and passes through \( x = 0 \) at \( t = 0 \). The two diagonals describe the paths of photons traveling at speed \( c \) in the \( \pm \hat{x} \) directions.

According to Postulate 1, if we change the velocity of our reference frame by “boosting” along the \( x \) axis, the speed of light will still be \( c \), meaning that in the new reference frame, the paths of photons will still have slope \( \pm 1 \). If, as per Postulate 2, the transformation is linear, these diagonals will be eigenvectors of the transformation. Figure 1b rotates the \((ct, x)\) coordinates by 45° to the \((\eta, \xi)\) basis that diagonalizes the boost. In this basis, a boost in velocity amounts to applying the linear transformation:

\[ \begin{pmatrix} \eta' \\ \xi' \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix} \] (1)

III. APPLICATIONS

Figure 2 illustrates the effect of a boost and gives a geometrical picture from which the velocity addition formula, length contraction, time dilation, the invariant interval, and the relativistic Doppler effect will be derived.
III.1. Velocity Addition

If one particle is traveling at \( v_1 \), another at \( v_2 \), the relative speed as seen from the reference frame of particle 1 will not generally be \( v_2 - v_1 \). To get the correct result, take a triangle with one vertex at the origin, one at the point \((1, m_1)\), and one at the point \((1, m_2)\), as illustrated by the blue lines in Figure 2a. Next, boost to a frame where \( m_1 \) is along the diagonal, corresponding to the rest frame of particle 1 (Figure 2b).

In this frame, the point \((1, m_2)\) transforms to:

\[
\sqrt{\frac{1 + \beta_2}{1 - \beta_1}} \left( \frac{1}{m'} \right) = \sqrt{\frac{1 + \beta_2}{1 - \beta_1}} \left( \frac{1}{1 + \beta_1} m_2 \right).
\]

Solving for \( m' \) gives:

\[
\beta' = \frac{\beta_2 - \beta_1}{1 - \beta_1 \beta_2},
\]

as the relative velocity. The velocity addition formula from Special Relativity for two particles moving along the same axis follows from taking \( \beta_1 \rightarrow -\beta_1 \).

III.2. Length Contraction

Figure 2 also illustrates length contraction. A solid object at rest traces out a diagonal ribbon parallel to the slope \( m = 1 \) time axis. Let one edge be at \( x = 0 \) and the other be at \( x = -L_P \). This gives \( \xi = \eta \) and \( \xi = \eta - \sqrt{2}L_P \), corresponding to the top and bottom green lines in Figure 2a, respectively. When the reference frame is boosted by \( v \) in Figure 2b, the edges are described by:

\[
\xi' = \frac{1 - \beta}{1 + \beta} \eta' \quad \text{and} \quad \xi' = \frac{1 - \beta}{1 + \beta} \eta' - \sqrt{1 + \beta} \times \sqrt{2}L_P.
\]

Physical length is measured at constant time. This is shown by the red arrows, which mark the separation, as measured along the \( x \) and \( x' \) axes, between the top and bottom green lines in the stationary and boosted frames. The line \( \xi' = -\eta' \) in Figure 2b intersects the top green line at \((0, 0)\) and the bottom green line at \( \sqrt{1 - \beta^2} \times \frac{L_P}{\sqrt{2}}(1, -1) \). The distance between these points is \( \sqrt{1 - \beta^2}L_P \). The length of the object in the moving frame is thus contracted: \( L' = \frac{L_P}{\gamma} \), compared to the proper length \( L_P \) in the object’s rest frame.

III.3. Time Dilation

Proper time is the time between two events at the same \( x \). This corresponds to the distance between the origin and the brown cross in Figure 2a. The points \((0, 0)\) and \( \frac{ct}{\sqrt{2}}(1, 1) \) transform to:

\[
\left( \begin{array}{c} 0 \\ 0 \end{array} \right) \quad \text{and} \quad \frac{ct}{\sqrt{2}} \left( \begin{array}{c} \sqrt{1 + \beta} \\ \sqrt{1 + \beta} \end{array} \right)
\]

in Figure 2b. Here, the time separation is the distance between the \( x' \) axis and the dashed brown line:

\[
ct' = \frac{\eta' + \xi'}{\sqrt{2}} = \frac{ct}{\sqrt{1 - \beta^2}} \rightarrow t' = \gamma t_P.
\]

This result is known as time dilation. The time between two events is longer in a reference frame where those events occur at different \( x \) positions.

III.4. The Invariant Interval

The fact that area elements are invariant tells us that:

\[
d\eta \times d\xi = \frac{1}{2} (cdt - dx) \times (cdt + dx)
\]

\[
\propto dx^2 - c^2 dt^2
\]

is invariant under boosts. Plotting the coordinates of two events, \( A \) and \( B \), in the \((\eta, \xi)\) plane, the area of a rectangular envelope with these two events at opposite corners (the light blue regions in Figure 2) is proportional to the invariant interval between these events: \( \Delta s^2 = \Delta x^2 - c^2 \Delta t^2 \).

III.5. Relativistic Doppler Effect

In the \((\eta, \xi)\) plane, horizontal lines correspond to photons traveling in the \(-\hat{x}\) direction, while vertical lines correspond to photons traveling in the \(+\hat{x}\) direction. The frequency observed by a person at \( x = 0 \) is inversely proportional to the distance between intersections of these gridlines and the time axis (the \( m = 1 \) diagonals shown in gray in Figure 2).

Since a boost in the \(+\hat{x}\) direction stretches \( \eta \) by \( \sqrt{1 + \beta^2} / (1 - \beta) \), the vertical gridlines in the boosted frame are further apart than in the rest frame. The frequency of light moving in the \(+\hat{x}\) direction is thus redshifted by a factor of \( \sqrt{1 - \beta^2} / (1 - \beta) \) as the observer moves away from the source.

Since the horizontal gridlines are closer together in the boosted frame, the frequency of light moving in the \(-\hat{x}\) direction is blue shifted by a factor of \( \sqrt{1 + \beta^2} / (1 - \beta) \).