A Conformal Basis for Scattering Amplitudes

SABRINA GONZALEZ PASTERSKI

Outline

- Why are we interested in superrotations?
- Which scattering basis do they prefer?
- What is the ‘conformal basis’?
- How can we transform amplitudes we know to this basis?
- Can we expect ‘standard’ CFT correlation functions?
- Does this analysis teach us something more about IR phyiscs?

Motivation

- What can IR physics teach us about gravitational scattering?

More Symmetries $\Rightarrow$ More Constraints

The **asymptotic symmetry group** of *asymptotically flat* spacetimes is much larger than Poincare

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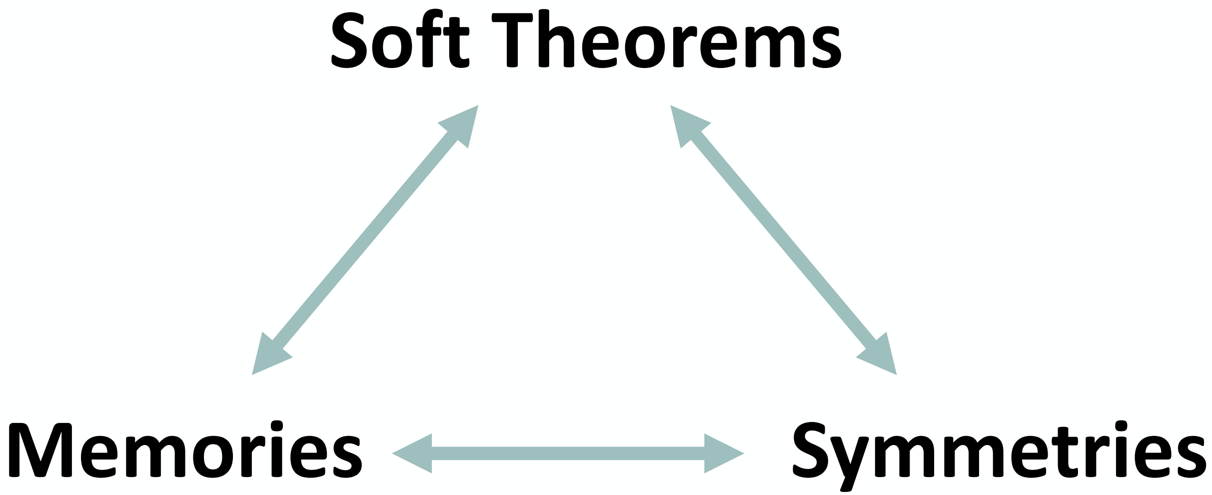
The **asymptotic symmetry group** of *asymptotically flat* spacetimes is much larger than Poincare

- Does this enhanced symmetry algebra hint at a celestial sphere holographic dual?

A Triangle of Relations

- What can IR physics teach us about gravitational scattering?

There exists a generic pattern of connections between asymptotic symmetries, soft theorems, and memory effects…



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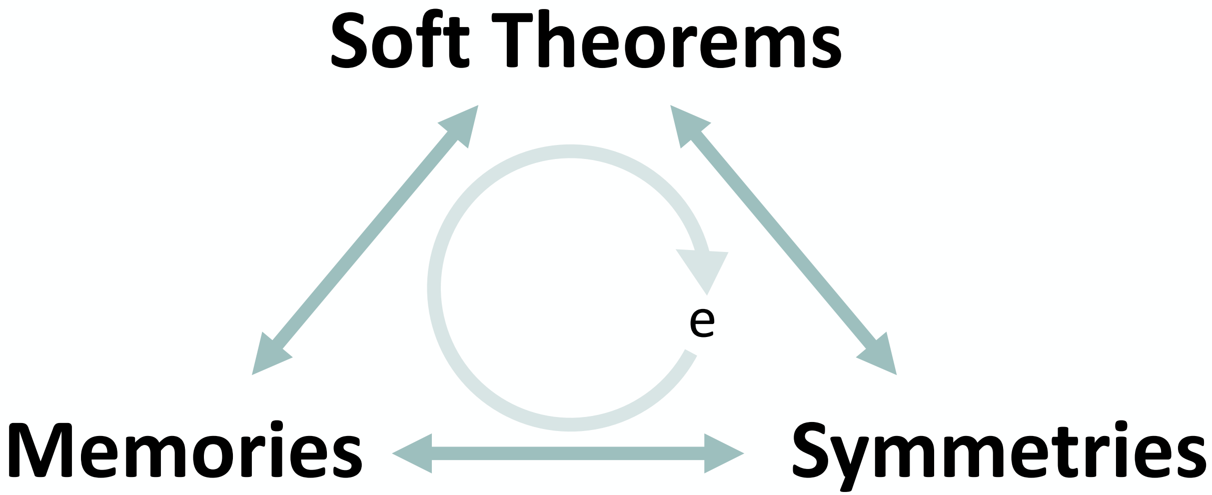

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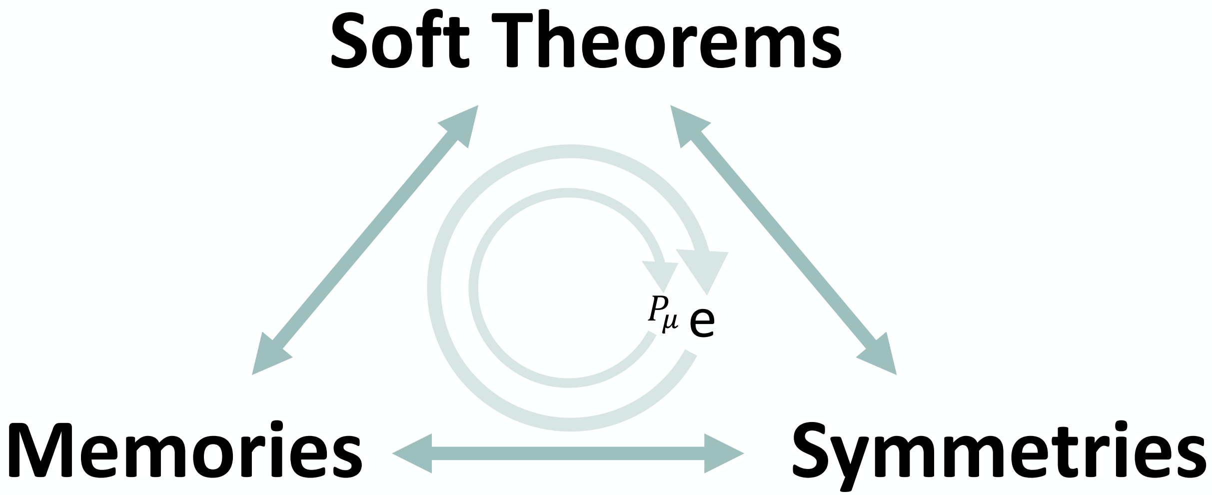

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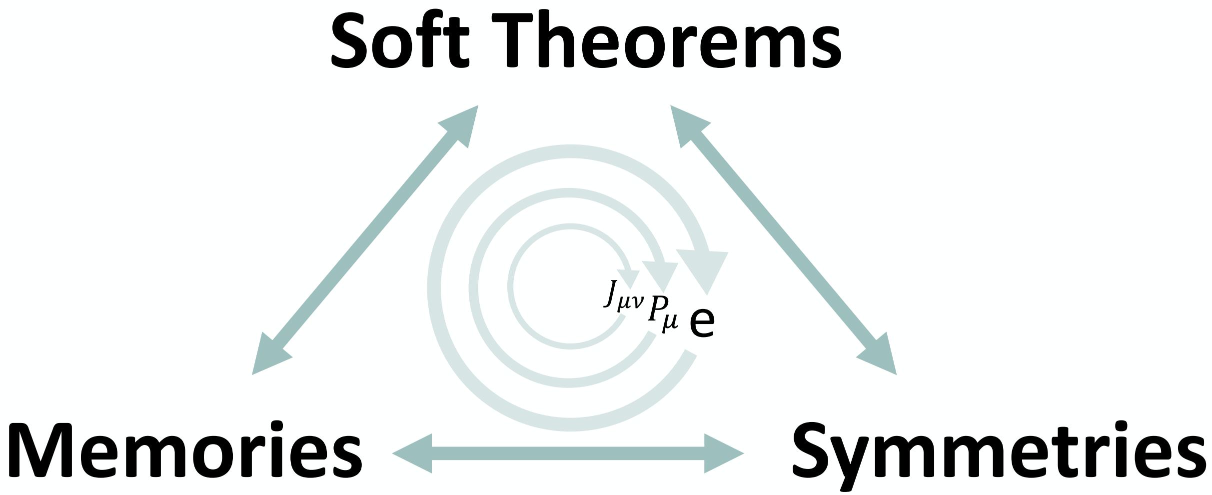

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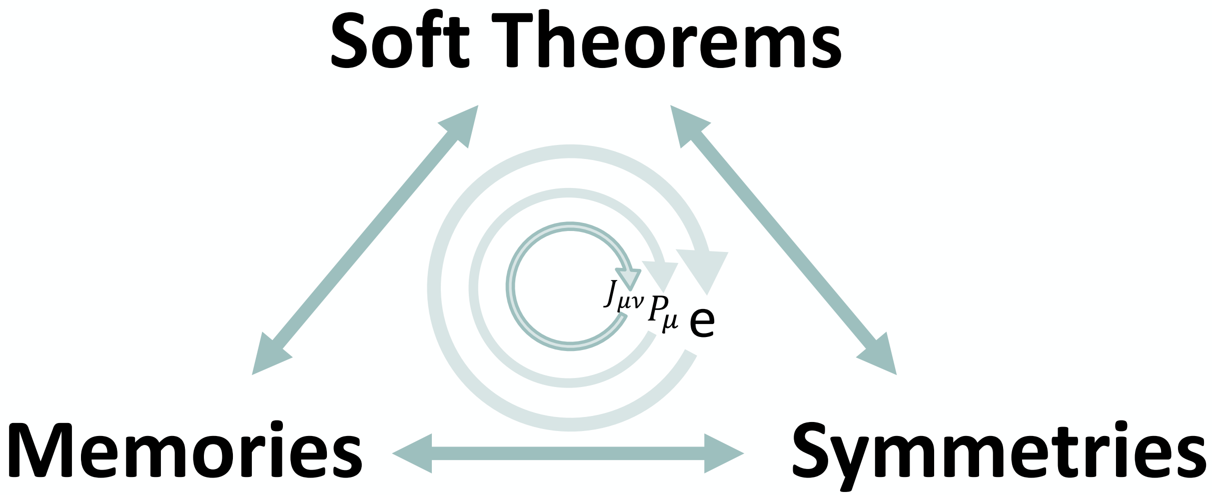
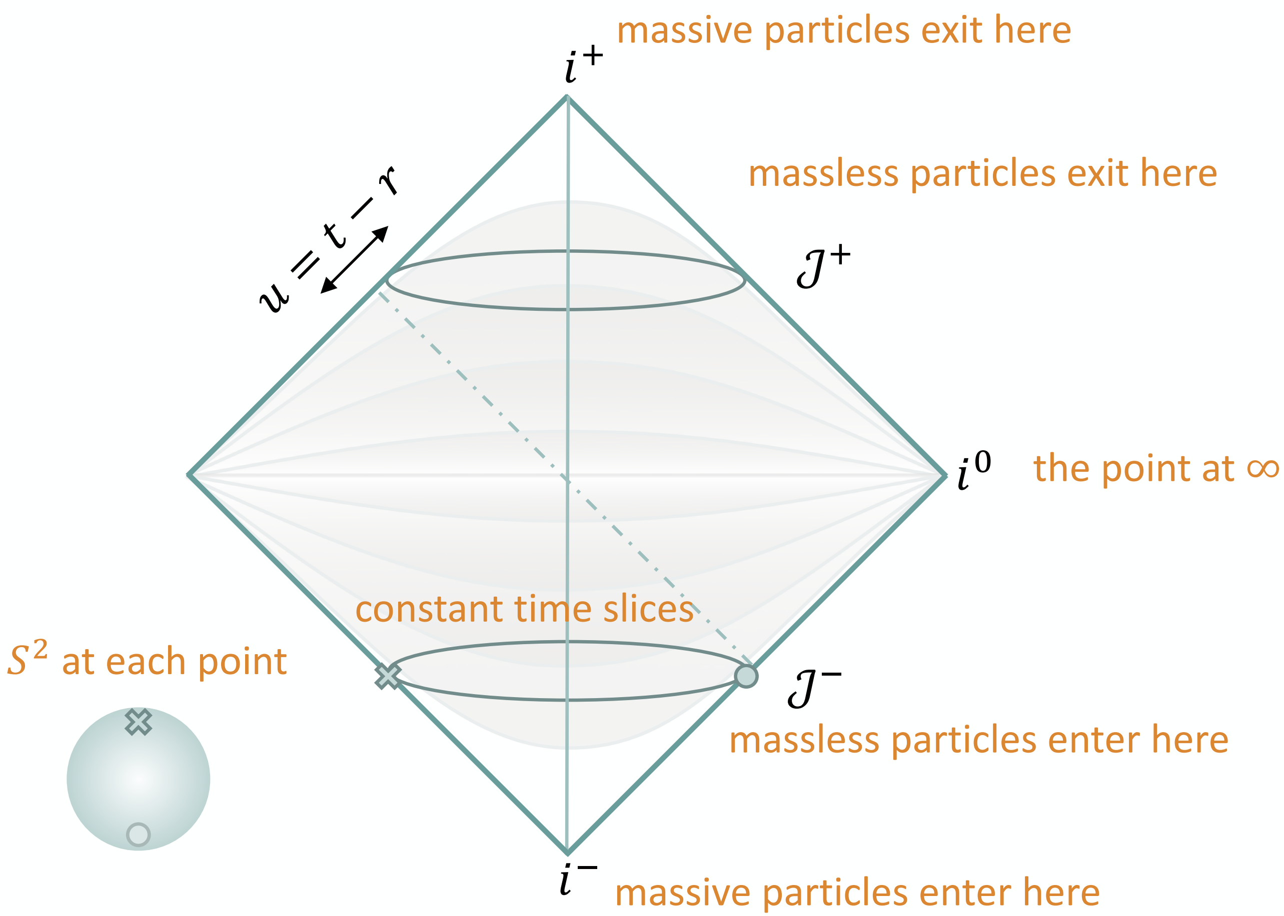


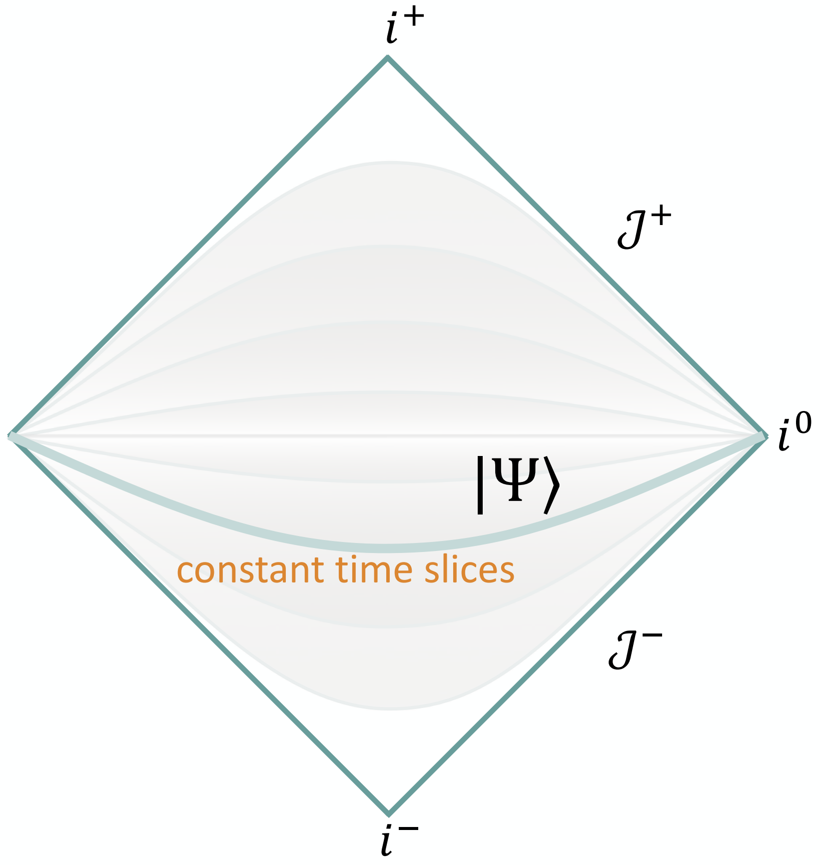
In this manner a brand new iteration was completed corresponding to ***superrotations***. This iteration is related to a generalization of Lorentz transformations and has motivated looking at $\mathcal{S}$-matrix elements in a new basis with definite SL(2,C) weights

A Simple Example

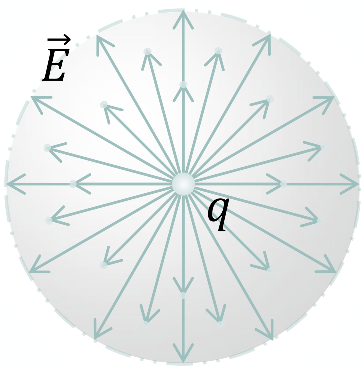


- Consider the conformal compactification of Minkowski spacetime

A Simple Example

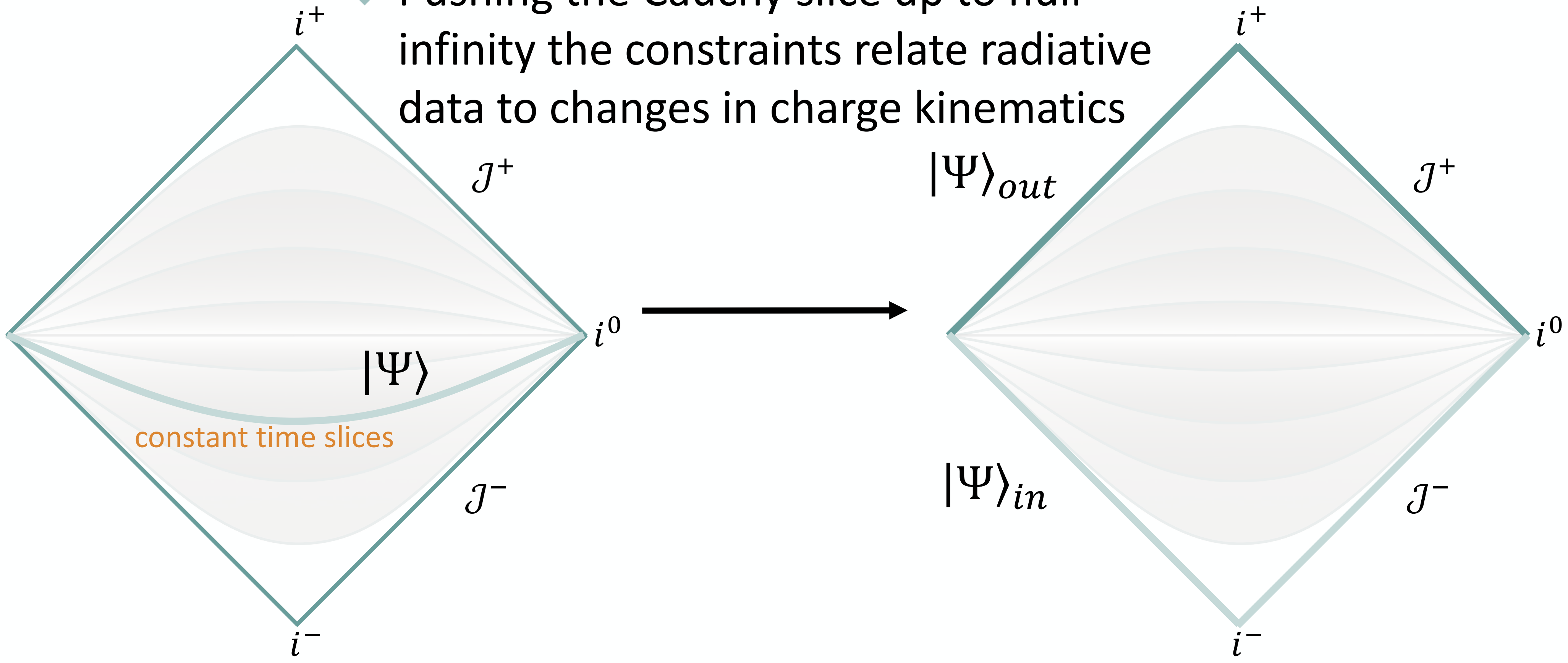


- In gauge theories there are constraints that need to be satisfied for the initial data on a Cauchy slice



A Simple Example

- Pushing the Cauchy slice up to null infinity the constraints relate radiative data to changes in charge kinematics



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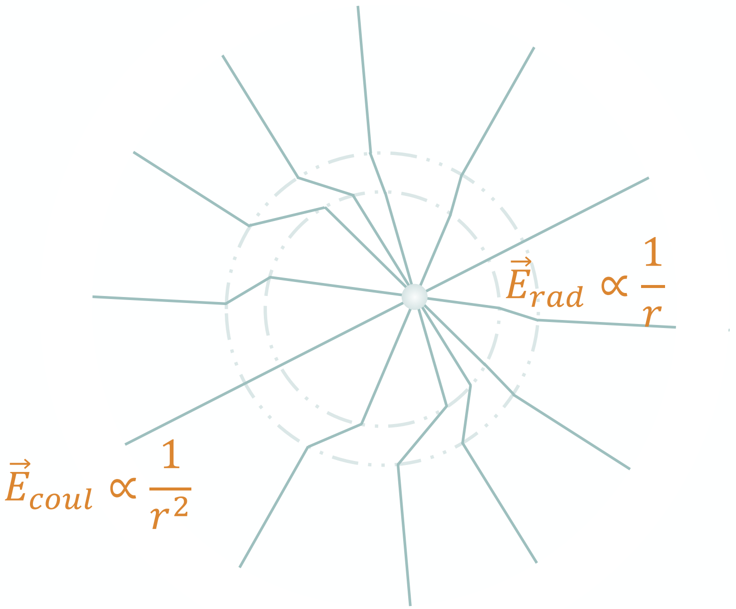
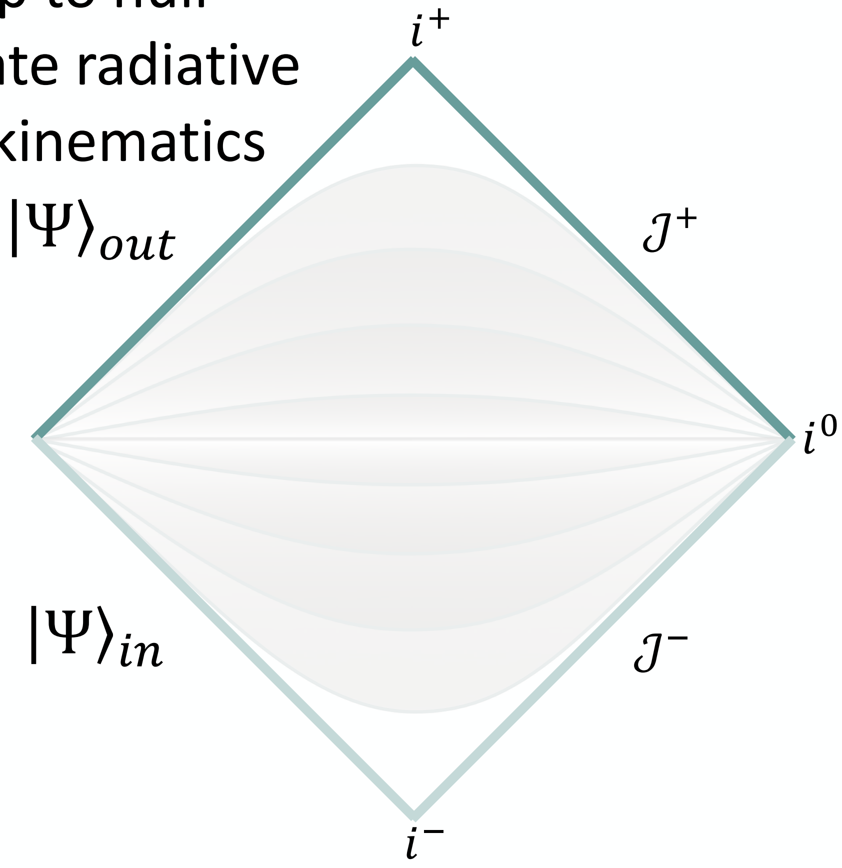




A Simple Example

Some more details:

- Radial Expansion:

$$\mathcal{A}_z(r,u,z,\bar{z}) = A_z(u,z,\bar{z}) + \sum_{n=1}^{\infty} \frac{A_z^{(n)}(u,z,\bar{z})}{r^n}$$

$$\mathcal{A}_u(r,u,z,\bar{z}) = \frac{1}{r}A_u(u,z,\bar{z}) + \sum_{n=1}^{\infty} \frac{A_u^{(n)}(u,z,\bar{z})}{r^{n+1}}$$

$$F_{ur} = A_u$$
$$F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z$$
$$F_{uz} = \partial_u A_z$$

- ASG that preserves this expansion:

$$\delta_\epsilon A_z(u,z,\bar{z}) = \partial_z \epsilon(z,\bar{z})$$

- Mode Expansion:

$$\mathcal{A}_\mu(x) = e \sum_{\alpha=\pm} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_q} \left[\epsilon_\mu^{\alpha^*}(\vec{q}) a_\alpha(\vec{q}) e^{iq\cdot x} + \epsilon_\mu^{\alpha}(\vec{q}) a_\alpha(\vec{q})^\dagger e^{-iq\cdot x} \right]$$

- Constraint Equation:

$$\partial_u A_u = \partial_u (D^z A_z + D^{\bar{z}} A_{\bar{z}}) + e^2 j_u$$

[arXiv:1407.3789]

Coordinate Conventions:

$$ds^2 = -du^2 - 2dudr + 2r^2 \gamma_{z\bar{z}} dz d\bar{z}$$

$$z = e^{i\phi} \tan\frac{\theta}{2} \qquad \gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$$

A Simple Example

Two key points:

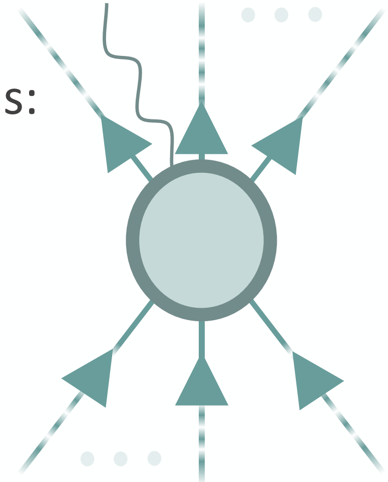
- Saddle point at large r picks out a gauge boson momentum pointing in the same direction as where an observer near null infinity would detect it. As a result, one ends up with a mode expansion where the angular integral localizes, and (u,ω) remain as Fourier conjugates.

$$e^{iq\cdot x} = e^{-i\omega u - i\omega r(1-\hat{q}\cdot\hat{x})} \qquad \Rightarrow \qquad A_z(u,z,\bar{z}) = -\frac{i}{8\pi^2}\frac{\sqrt{2}e}{1+z\bar{z}}\int_0^\infty d\omega\left[a_+(\omega\hat{x})e^{-i\omega u} - a_-(\omega\hat{x})^\dagger e^{i\omega u}\right]$$

- $\int du$ picks out $\omega \to 0$. As such we can relate the soft factors to the constraint equations: Fourier transform of a pole $\frac{1}{\omega}$ is a step function

$$S^{(0)-} = \sum_k eQ_k \frac{p_k\cdot\epsilon^-}{p_k\cdot q}$$

$$\langle z_{n+1}, z_{n+2}, \ldots|a_-(q)\mathcal{S}|z_1, z_2, \ldots\rangle = S^{(0)-}\langle z_{n+1}, z_{n+2}, \ldots|\mathcal{S}|z_1, z_2, \ldots\rangle + \mathcal{O}(1)$$

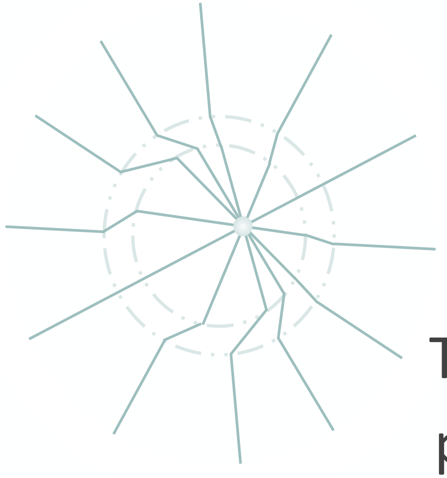


[arXiv:1407.3789, arXiv:1505.00716]

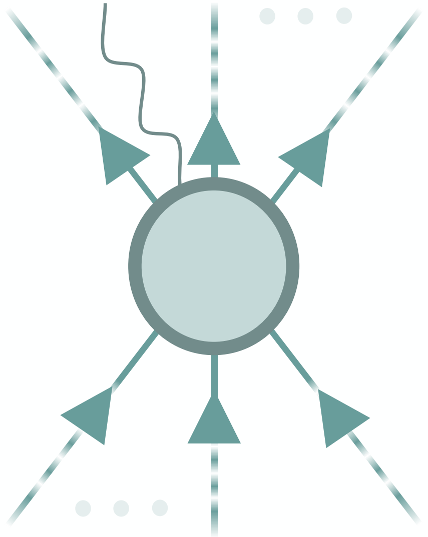
A Simple Example

Integrate the constraint equation along u

$$E_r = \frac{Q}{4\pi r^2}\frac{1}{\gamma^2(1-\vec{\beta}\cdot\hat{n})^2}$$



$$\Delta A_u = 2D^z \Delta A_z + e^2 \int du j_u$$



$$-\frac{e}{4\pi}\lim_{\omega\to 0}\omega[D^z\hat{\epsilon}_z^{*+}S_p^{(0)+} + D^{\bar{z}}\hat{\epsilon}_{\bar{z}}^{*-}S_p^{(0)-}] \quad = -e^2\frac{Q}{4\pi}\frac{1}{\gamma^2(1-\vec{\beta}\cdot\hat{n})^2}$$

The soft factor indicates that typical scattering processes will produce a nonzero u integrated electric field.

Some Conventions:

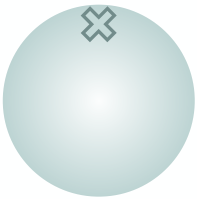
$$p^\mu = m\gamma(1,\vec{\beta}) \qquad S_p^{(0)\pm} = eQ\frac{p\cdot\epsilon^\pm}{p\cdot q}$$

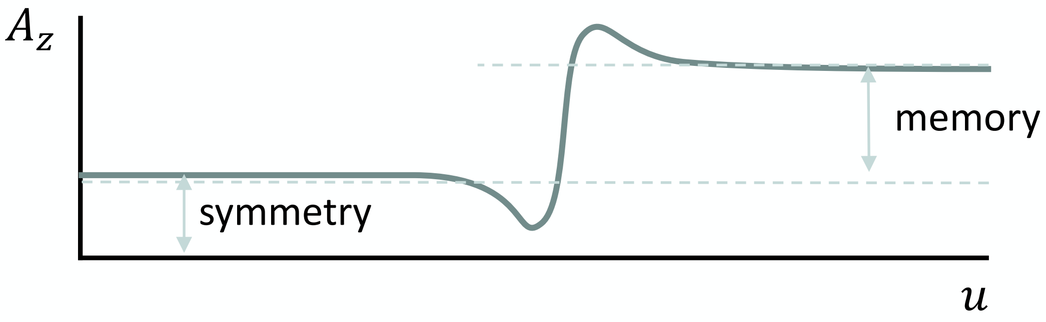
$$\Delta A_z = -\frac{e}{4\pi}\hat{\epsilon}_z^{*+}\omega S^{(0)+}$$

[**arXiv:1505.00716**]

A Simple Example

- Upshot: The residue of the Weinberg pole indicates a nonzero value for certain low-energy radiation observables aka **“memory effects”**
- Since setting these modes to zero would trivialize the allowed scattering events, we get with this class of boundary conditions a larger class of gauge transformations that preserve the radial order of the falloffs while shifting the boundary values aka **“large gauge transformations”**





A Triangle of Relations

relate $\mathcal{S}$-matrix elements for states with and without
extra soft gauge particle

Soft Theorems

Fourier transform:
long time ↔ low energy

Ward identities
[arXiv:1308.0589 arXiv:1312.2229]

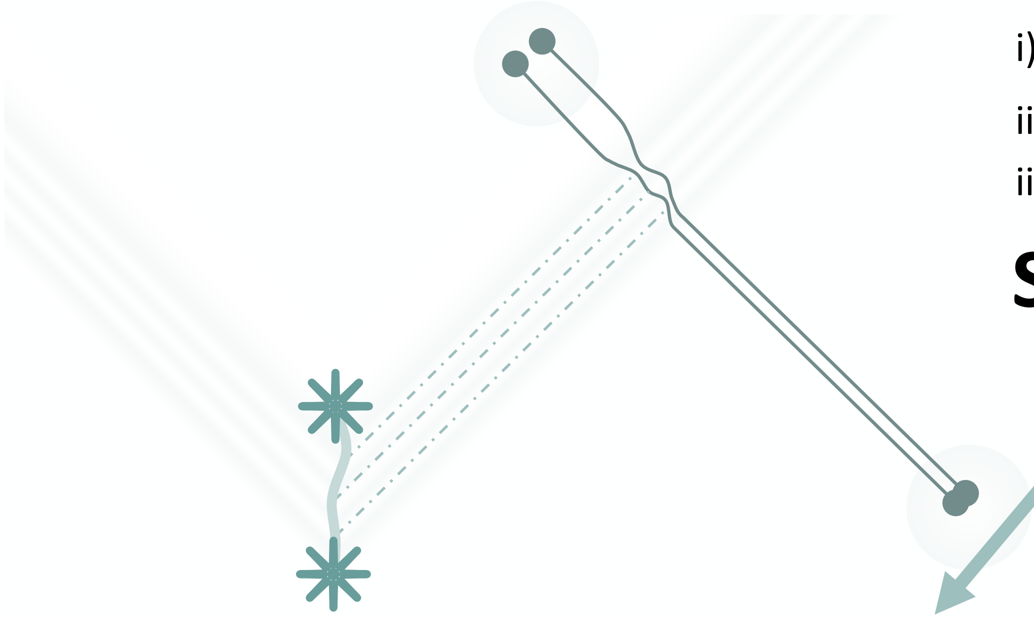
Memories ↔ **Symmetries**

non-zero net effects in a typical scattering process
forces us to have asymptotic behavior that allows them,
these extra symmetries then act non-trivially

step (net change)
vs baseline (starting point)

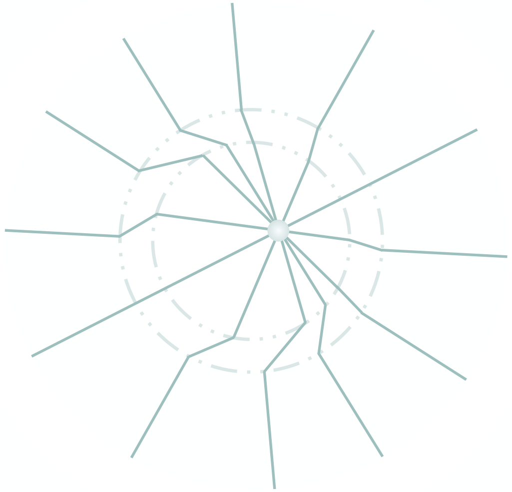
what we're after:
More Symmetries ⇒ More Constraints on $\mathcal{S}$-matrix

A Triangle of Relations



i) Weinberg – photon O($\frac{1}{\omega}$)
ii) Weinberg – graviton O($\frac{1}{\omega}$)
iii) Cachazo & Strominger – graviton O(1)

Soft Theorems



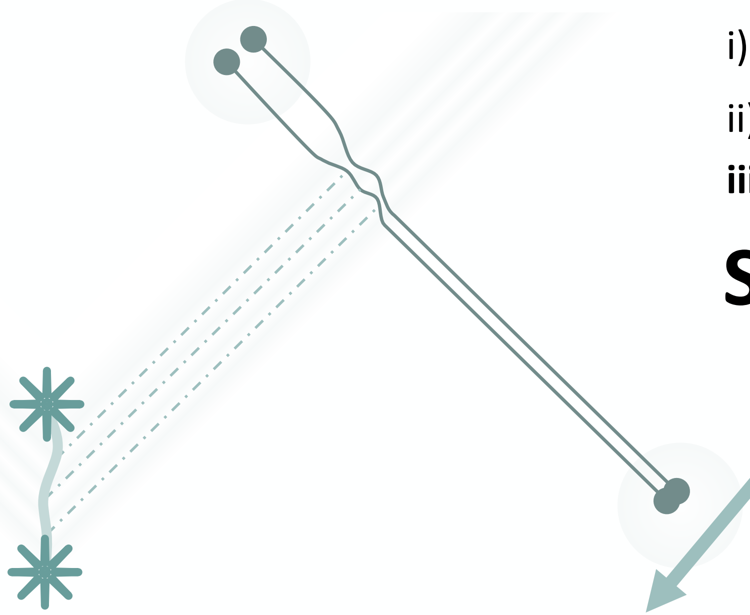
Memories

i) Liénard-Wiechert / Bieri & Garfinkle
ii) Zeldovich & Polnarev / Christodoulou
iii) Pasterski, Strominger, & Zhiboedov

Symmetries

(global)	(asymptotic)
i) e-charge	large U(1)
ii) P_μ	supertranslations
iii) $J_{\mu\nu}$	superrotations

A Triangle of Relations



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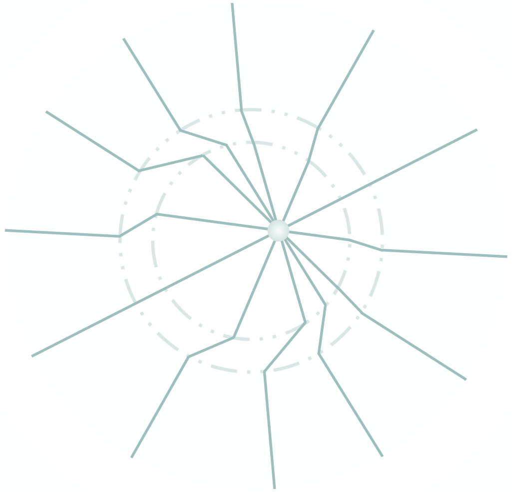

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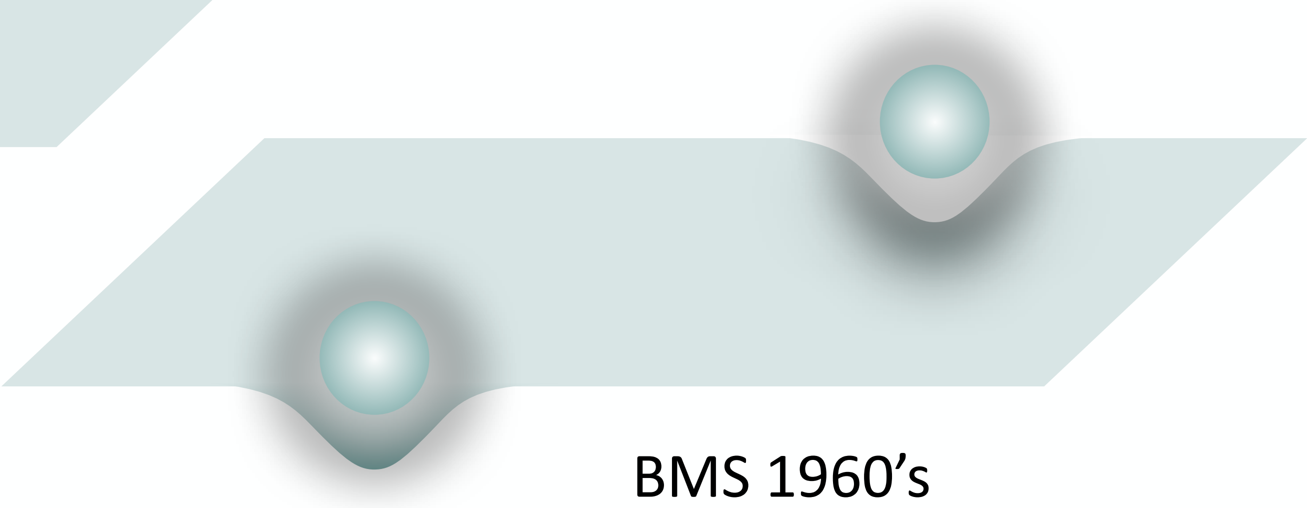
Asymptotically Flat Spacetimes

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

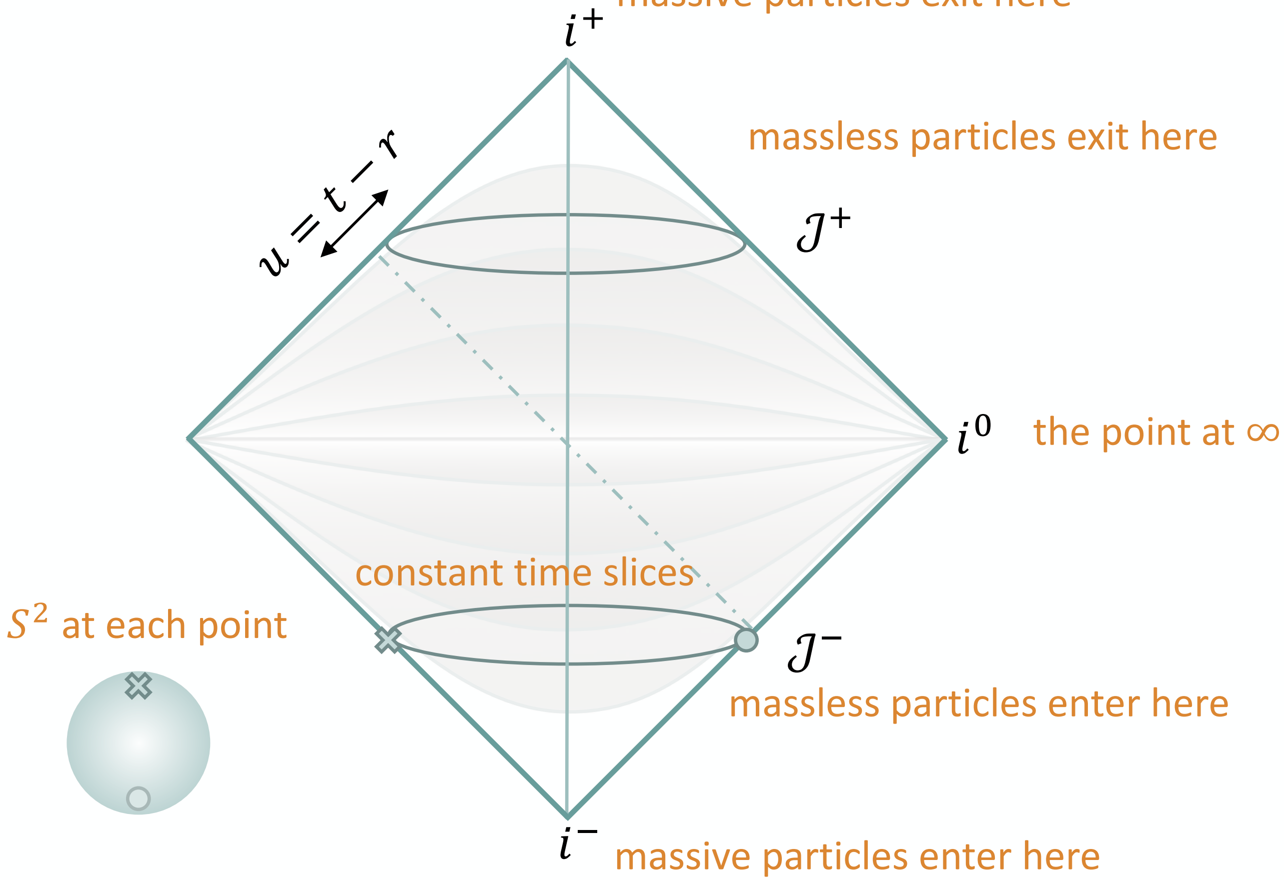
- Want to consider non-trivial gravitational backgrounds that are ``close" to being flat



BMS 1960's

  - Approach flat spacetime far away from sources

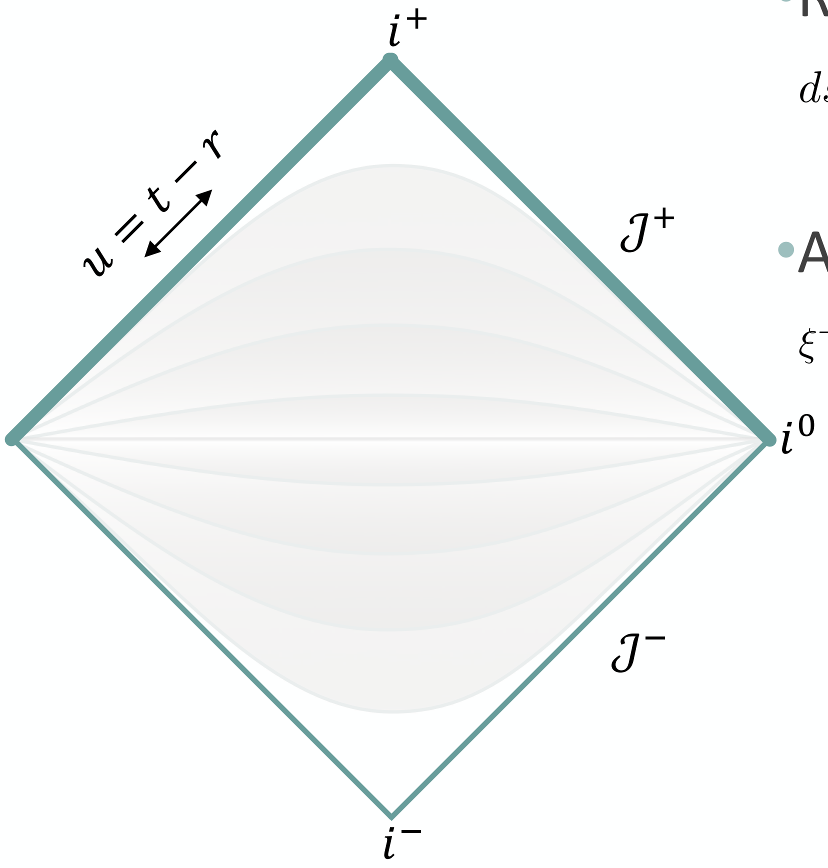
Asymptotically Flat Spacetimes



- Interested in set of diffeomorphisms that preserve class of asymptotically flat metrics, characterized by radial fall-off near null infinity

- $$\text{ASG} = \frac{\text{allowed gauge symmetries}}{\text{trivial gauge symmetries}}$$

Asymptotically Flat Spacetimes



- Radial Expansion:

$$\begin{aligned} ds^2 \quad &= -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} + 2\tfrac{m_B}{r}du^2 \\ &+\big(rC_{zz}dz^2 + D^zC_{zz}dudz + \tfrac{1}{r}(\tfrac{4}{3}N_z - \tfrac{1}{4}\partial_z(C_{zz}C^{zz}))dudz + c.c.\big) + \ldots \end{aligned}$$

- ASG that preserves this expansion:

$$\begin{aligned} \xi^+ =&(1+\frac{u}{2r})Y^{+z}\partial_z - \frac{u}{2r}D^{\bar{z}}D_zY^{+z}\partial_{\bar{z}} - \frac{1}{2}(u+r)D_zY^{+z}\partial_r + \frac{u}{2}D_zY^{+z}\partial_u + c.c \\ &+ f^+\partial_u - \frac{1}{r}(D^zf^+\partial_z + D^{\bar{z}}f^+\partial_{\bar{z}}) + D^zD_zf^+\partial_r \end{aligned}$$

Coordinate Conventions:

$$z = e^{i\phi}\tan\frac{\theta}{2} \qquad \gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$$

$$f^+ = f^+(z,\bar{z}) \qquad \partial_{\bar{z}}Y^{+z} = 0$$

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&+ f^+\partial_u - \frac{1}{r}(D^zf^+\partial_z + D^{\bar{z}}f^+\partial_{\bar{z}}) + D^zD_zf^+\partial_r
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$$

Superrotation Charge

- We can demonstrate a semiclassical Ward identity for superrotations using the subleading soft graviton theorem [**arXiv:1406.3312**].

$$< out|Q^+[Y]\mathcal{S} - \mathcal{S}Q^-[Y]|in >= 0$$

$$8\pi G Q^+[Y] = \int du \int d^2z \sqrt{\gamma}\partial_u[-uY^A D_A m_B + Y^A N_A + ...]$$

$$\partial_u m_B = \frac{1}{4}\partial_u\left[D_z^2 C^{zz} + D_{\bar{z}}^2 C^{\bar{z}\bar{z}}\right] - T_{uu}$$

$$\partial_u N_z = \frac{1}{4}\partial_z\left[D_z^2 C^{zz} - D_{\bar{z}}^2 C^{\bar{z}\bar{z}}\right] + \partial_z m_B - T_{uz}$$

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$$Q^+[Y] = Q_S^+[Y] + Q_H^+[Y]$$

$$Q_S^+[Y] = \frac{1}{2}\int_{\mathcal{I}^+} du d^2 z D_z^3 Y^z u \partial_u C_{\bar{z}}^z$$

$$Q_H^+[Y] = \lim_{\Sigma \to \mathcal{I}^+} \int_\Sigma d\Sigma\, \xi^\mu n_\Sigma^\nu T_{\mu\nu}^M$$

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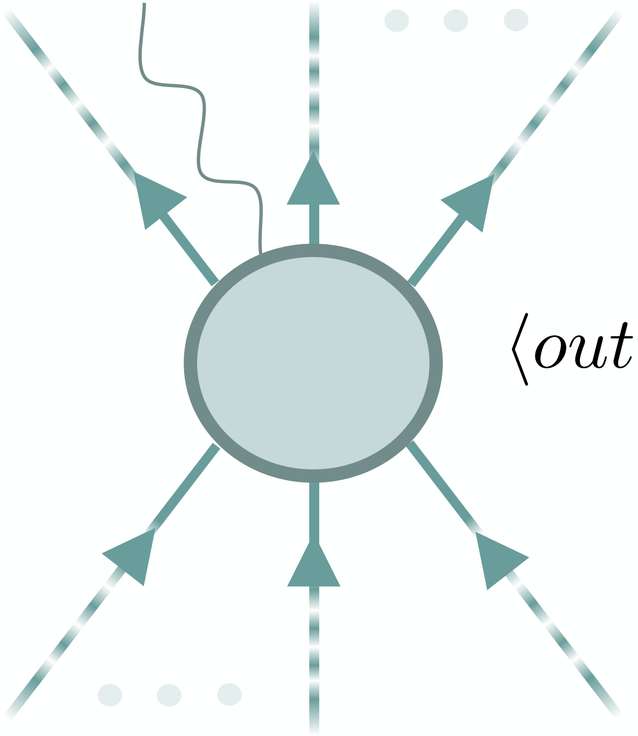

as Fourier mode of field operator

$$\boxed{\begin{array}{c} e^{iq\cdot x} = e^{-i\omega u - i\omega r(1-\hat{q}\cdot\hat{x})} \\ \hat{q}_{pos} \Leftrightarrow \hat{q}_{mom} \quad \& \quad \lim_{\omega\to 0} \Leftrightarrow \int du \end{array}}$$

$$\langle out|a_-(q)\mathcal{S}|in\rangle = \left(S^{(0)-} + S^{(1)-}\right)\langle out|\mathcal{S}|in\rangle + \mathcal{O}(\omega)$$

$$S^{(0)-} = \sum_k \frac{(p_k\cdot\epsilon^-)^2}{p_k\cdot q} \qquad S^{(1)-} = -i\sum_k \frac{p_{k\mu}\epsilon^{-\mu\nu}q^\lambda J_{k\lambda\nu}}{p_k\cdot q}$$

Superrotation Charge

- Looking again at the superrotation vector field near null infinity, we notice we have two copies of the Witt algebra:

$$\xi^+|_{\mathcal{J}^+} = Y^{+z}\partial_z + \frac{u}{2}D_z Y^{+z}\partial_u + c.c.$$

- Moreover, for a particular choice of $Y^z \sim \frac{1}{z-w}$ we find that the soft part of the charge takes the form of a putative 2D stress tensor [arXiv:1609.00282] .

$$T_{zz} \equiv \frac{i}{8\pi G}\int d^2w \frac{1}{z-w} D_w^2 D^{\bar{w}} \int du u \partial_u C_{\bar{w}\bar{w}}$$

$$\langle T_{zz}\mathcal{O}_1 \cdots \mathcal{O}_n\rangle = \sum_{k=1}^{n}\left[\frac{h_k}{(z-z_k)^2} + \frac{\Gamma^{z_k}_{z_k z_k}}{z-z_k}h_k + \frac{1}{z-z_k}\left(\partial_{z_k} - |s_k|\Omega_{z_k}\right)\right]\langle\mathcal{O}_1\cdots\mathcal{O}_n\rangle$$

Weight Conventions:

$$h = \frac{1}{2}(s+1+iE_R) \quad \bar{h} = \frac{1}{2}(-s+1+iE_R)$$

$$\Delta = h + \bar{h} \quad s = h - \bar{h}$$

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We need Rindler energy eigenstates!

Constructing A Conformal-Primary Basis

- Using that the Lorentz group SO(1,d+1) in $\boldsymbol{R}^{1,d+1}$ acts as the conformal group on $\boldsymbol{R}^{d}$ define the *massive scalar conformal primary wavefunction* to:

  - satisfy the (d+2)-dimensional massive Klein-Gordon equation of mass *m*:

$$\left(\frac{\partial}{\partial X^{\nu}}\frac{\partial}{\partial X_{\nu}} - m^2\right)\phi_{\Delta}(X^{\mu};\vec{w}) = 0$$

  - transform covariantly as a scalar conformal primary operator in d dimensions under an SO(1,d+1) transformation:

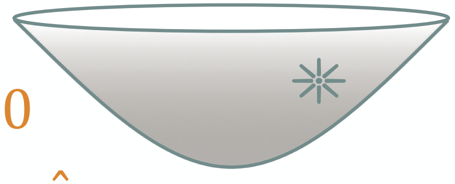
$$\phi_{\Delta}\left(\Lambda^{\mu}{}_{\nu}X^{\nu};\vec{w}\,'(\vec{w})\right) = \left|\frac{\partial\vec{w}\,'}{\partial\vec{w}}\right|^{-\Delta/d}\phi_{\Delta}(X^{\mu};\vec{w})$$

[arXiv:1705.01027]

Constructing A Conformal-Primary Basis

$$ds^2_{H_{d+1}} = \frac{dy^2 + d\vec{z} \cdot d\vec{z}}{y^2}$$

$\hat{p}$



$m \neq 0$

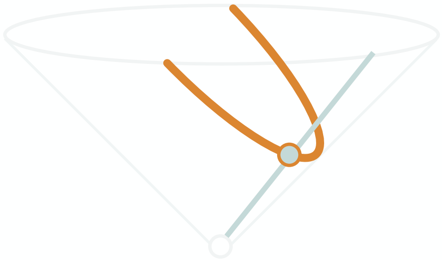
$p = m\hat{p}$

$$\hat{p}(y, \vec{z}) = \left(\frac{1 + y^2 + |\vec{z}|^2}{2y} \, , \, \frac{\vec{z}}{y} \, , \, \frac{1 - y^2 - |\vec{z}|^2}{2y} \right)$$

$$\hat{p}^\mu(y', \vec{z}\,') = \Lambda^\mu{}_\nu \hat{p}^\nu$$

q

$\vec{w} \in \boldsymbol{R}^d$



$$q^\mu(\vec{w}) = \left(1 + |\vec{w}|^2 \, , \, 2\vec{w} \, , \, 1 - |\vec{w}|^2 \right)$$

$$q^\mu(\vec{w}\,') = \left| \frac{\partial \vec{w}\,'}{\partial \vec{w}} \right|^{1/d} \Lambda^\mu{}_\nu q^\nu(\vec{w})$$

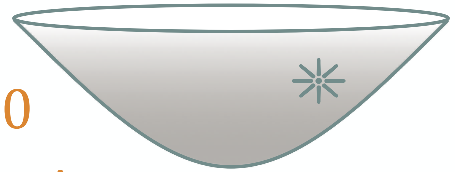
Constructing A Conformal-Primary Basis

$$G_\Delta(\hat p; q) = \frac{1}{(-\hat p\cdot q)^\Delta} = \left(\frac{y}{y^2+|\vec z-\vec w|^2}\right)^\Delta$$

$$G_\Delta(\hat p'; q') = \left|\frac{\partial \vec w\,'}{\partial \vec w}\right|^{-\Delta/d} G_\Delta(\hat p; q)$$

$$ds^2_{H_{d+1}} = \frac{dy^2 + d\vec z\cdot d\vec z}{y^2}$$

$\hat p$



$m \neq 0$

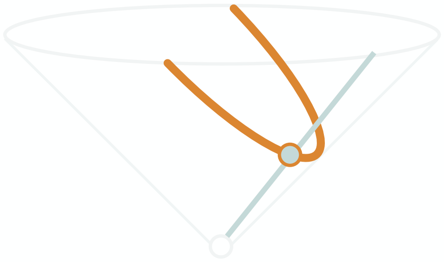
$p = m\hat p$

$$\hat p(y,\vec z) = \left(\frac{1+y^2+|\vec z|^2}{2y}, \frac{\vec z}{y}, \frac{1-y^2-|\vec z|^2}{2y}\right)$$

$$\hat p^\mu(y', \vec z\,') = \underline{\Lambda^\mu{}_\nu \hat p^\nu}$$

q

$\vec w \in \boldsymbol{R}^d$

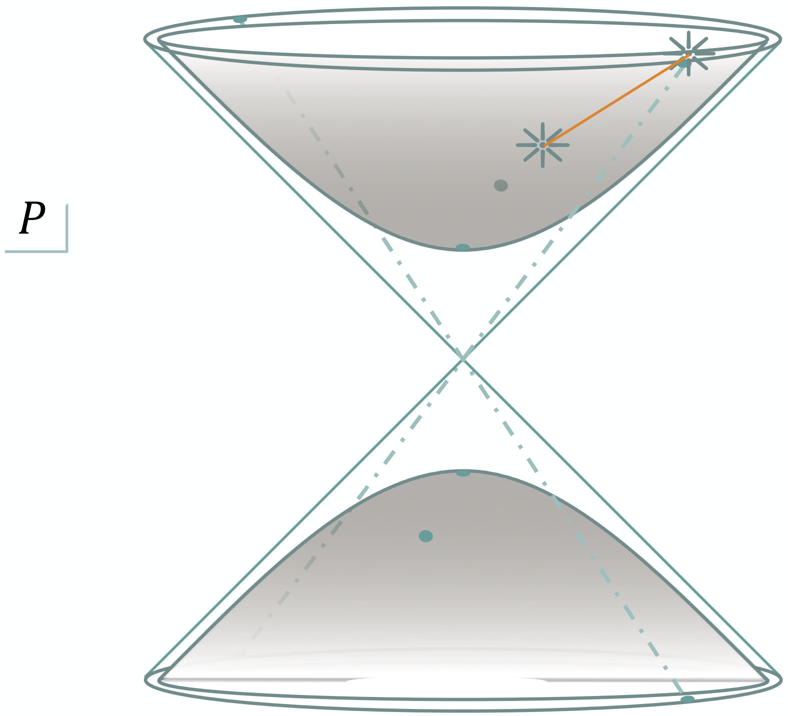


$$q^\mu(\vec w) = (1+|\vec w|^2, 2\vec w, 1-|\vec w|^2)$$

$$q^\mu(\vec w\,') = \underline{\left|\frac{\partial \vec w\,'}{\partial \vec w}\right|^{1/d} \Lambda^\mu{}_\nu q^\nu(\vec w)}$$

Constructing A Conformal-Primary Basis

- The desired properties are met by the convolution:



$$\phi^{\pm}_{\Delta}(X^{\mu};\vec{w}) = \int_{H_{d+1}} [d\hat{p}]\, G_{\Delta}(\hat{p};\vec{w})\, \exp\left[\, \pm i m \hat{p}\cdot X \,\right]$$

- Interpretation as bulk-to-boundary propagation in momentum space
- Have **plane wave** $\Rightarrow$ **highest-weight**, what about reverse?

Constructing A Conformal-Primary Basis

- The orthogonality conditions

$$\int_{-\infty}^{\infty} d\nu\, \mu(\nu) \int d^d\vec{w}\, G_{\frac{d}{2}+i\nu}(\hat{p}_1;\vec{w})G_{\frac{d}{2}-i\nu}(\hat{p}_2;\vec{w}) = \delta^{(d+1)}(\hat{p}_1,\hat{p}_2)$$

$$\int_{H_{d+1}} [d\hat{p}]\, G_{\frac{d}{2}+i\nu}(\hat{p};\vec{w}_1)G_{\frac{d}{2}+i\bar{\nu}}(\hat{p};\vec{w}_2) =$$

$$2\pi^{d+1}\frac{\Gamma(i\nu)\Gamma(-i\nu)}{\Gamma(\frac{d}{2}+i\nu)\Gamma(\frac{d}{2}-i\nu)}\delta(\nu+\bar{\nu})\delta^{(d)}(\vec{w}_1-\vec{w}_2) + 2\pi^{\frac{d}{2}+1}\frac{\Gamma(i\nu)}{\Gamma(\frac{d}{2}+i\nu)}\delta(\nu-\bar{\nu})\frac{1}{|\vec{w}_1-\vec{w}_2|^{2(\frac{d}{2}+i\nu)}}$$

$$\mu(\nu) = \frac{\Gamma(\frac{d}{2}+i\nu)\Gamma(\frac{d}{2}-i\nu)}{4\pi^{d+1}\Gamma(i\nu)\Gamma(-i\nu)}$$

[arXiv:1404.5625]

- Imply we can go in the opposite direction **highest-weight** $\Rightarrow$ **plane wave**

$$\boxed{e^{\pm i m \hat{p}\cdot X} = 2\int_0^{\infty} d\nu\, \mu(\nu) \int d^d\vec{w}\; G_{\frac{d}{2}-i\nu}(\hat{p};\vec{w})\; \phi^{\pm}_{\frac{d}{2}+i\nu}(X^\mu;\vec{w})}$$

$\Delta \in \frac{d}{2} + i\boldsymbol{R}_{\geq 0}$ $\times$

$\vec{w} \in \boldsymbol{R}^d$

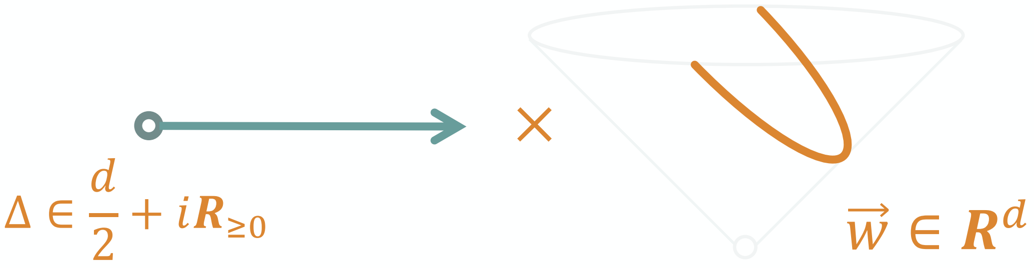
Constructing A Conformal-Primary Basis

- If we define the shadow for a scalar as

$$\widetilde{\mathcal{O}}_{\Delta}(\vec{w}) \equiv \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}\int d^d\vec{w}\,'\frac{1}{|\vec{w}-\vec{w}\,'|^{2(d-\Delta)}}\mathcal{O}_{\Delta}(\vec{w}\,')$$

- The action on our scalar wavefunctions shows linear dependence between weights Δ and $d-\Delta$

$$\widetilde{\phi^{\pm}_{\Delta}}(X;\vec{w}) = \phi^{\pm}_{d-\Delta}(X;\vec{w})$$



Constructing A Conformal-Primary Basis

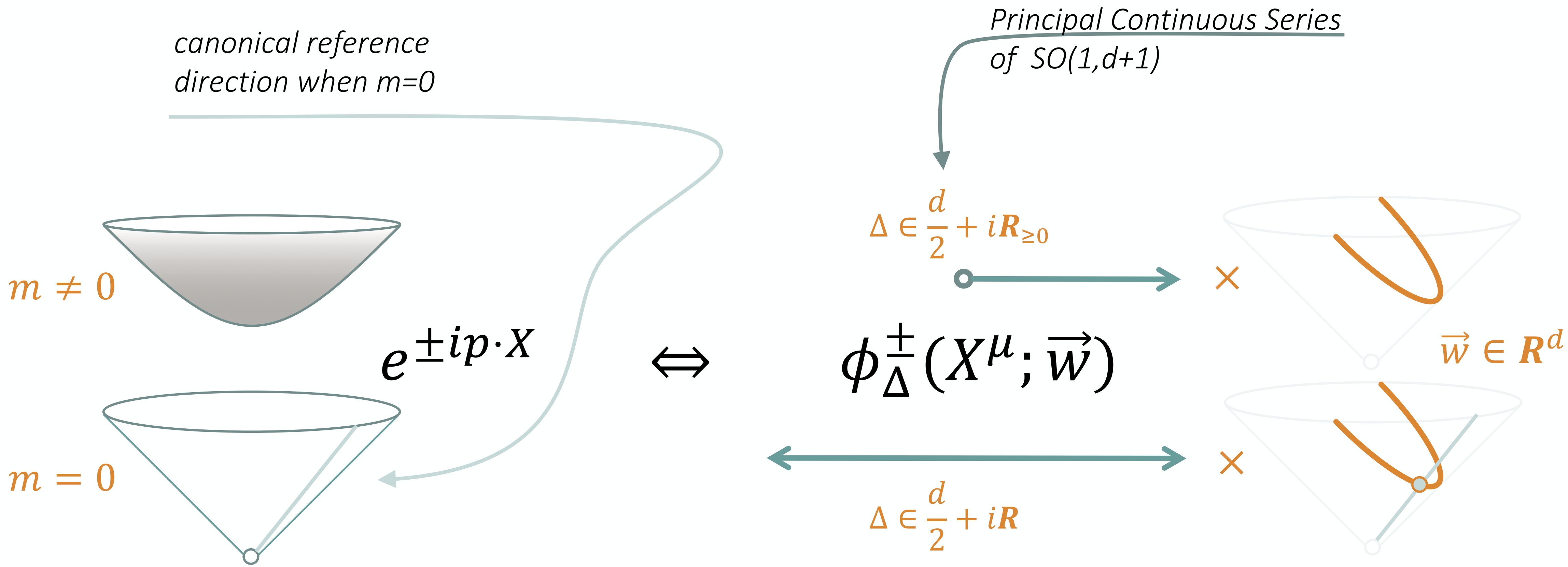
- By forming the combination $\omega = \frac{m}{2y}$ we can further use the boundary behavior of G_Δ to explore the massless analog:

$$G_\Delta(y, \vec{z}; \vec{w}) \underset{m \to 0}{\longrightarrow} \pi^{\frac{d}{2}} \frac{\Gamma(\Delta - \frac{d}{2})}{\Gamma(\Delta)} y^{d-\Delta} \delta^{(d)}(\vec{z} - \vec{w}) + \frac{y^\Delta}{|\vec{z} - \vec{w}|^{2\Delta}} + \cdots$$

- The first term results in a Mellin transform of the energy, in which the reference direction is the same as the momentum, and satisfies the desired properties of a massless conformal primary.

$$\boxed{\varphi^\pm_\Delta(X^\mu; \vec{w}) \equiv \int_0^\infty d\omega\, \omega^{\Delta-1}\, e^{\pm i\omega q\cdot X - \epsilon\omega} = \frac{(\mp i)^\Delta \Gamma(\Delta)}{(-q(\vec{w}) \cdot X \mp i\epsilon)^\Delta}}$$

Constructing A Conformal-Primary Basis



Constructing A Conformal-Primary Basis

- Photon

$$\left(\frac{\partial}{\partial X^\sigma}\frac{\partial}{\partial X_\sigma}\delta^\mu_\nu - \frac{\partial}{\partial X^\nu}\frac{\partial}{\partial X_\mu}\right) A^{\Delta\pm}_{\mu a}(X^\rho;\vec{w}) = 0$$

$$A^{\Delta\pm}_{\mu a}\left(\Lambda^\rho{}_\nu X^\nu;\vec{w}\,'(\vec{w})\right) = \frac{\partial w^b}{\partial w'^a}\left|\frac{\partial \vec{w}'}{\partial \vec{w}}\right|^{-(\Delta-1)/d} \Lambda_\mu{}^\sigma A^{\Delta\pm}_{\sigma b}(X^\rho;\vec{w})$$

- Graviton

$$\partial_\sigma\partial_\nu h^\sigma{}_{\mu;a_1a_2} + \partial_\sigma\partial_\mu h^\sigma{}_{\nu;a_1a_2} - \partial_\mu\partial_\nu h^\sigma{}_{\sigma;a_1a_2} - \partial^\rho\partial_\rho h_{\mu\nu;a_1a_2} = 0$$

$$h^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2} = h^{\Delta,\pm}_{\mu_2\mu_1;a_1a_2}\,,\qquad h^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2} = h^{\Delta,\pm}_{\mu_1\mu_2;a_2a_1}\,,\qquad \delta^{a_1a_2}h^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2} = 0$$

$$h^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2}\left(\Lambda^\rho{}_\nu X^\nu;\vec{w}\,'(\vec{w})\right) = \frac{\partial w^{b_1}}{\partial w'^{a_1}}\frac{\partial w^{b_2}}{\partial w'^{a_2}}\left|\frac{\partial \vec{w}'}{\partial \vec{w}}\right|^{-(\Delta-2)/d} \Lambda_{\mu_1}{}^{\sigma_1}\Lambda_{\mu_2}{}^{\sigma_2} h^{\Delta,\pm}_{\sigma_1\sigma_2;b_1b_2}(X^\rho;\vec{w})$$

$$\Delta \in \frac{d}{2} + i\boldsymbol{R} \qquad \times \qquad \vec{w} \in \boldsymbol{R}^d$$

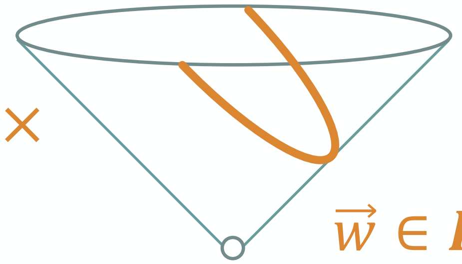
Constructing A Conformal-Primary Basis

- The shadow is linearly independent.
- Demanding conformal profile fixes residual gauge transformations but within gauge equivalence class can return to Mellin representative.

$$A^{\Delta,\pm}_{\mu a}(X^\mu;\vec{w}) = \frac{\partial_a q_\mu}{(-q\cdot X \mp i\epsilon)^{\Delta}} + \frac{\partial_a q\cdot X}{(-q\cdot X \mp i\epsilon)^{\Delta+1}}q_\mu$$

$$-\,const.\ \frac{\partial}{\partial X^\mu}\left(\frac{\partial_a q\cdot X}{(-q\cdot X \mp i\epsilon)^{\Delta}}\right)$$

$\Delta \in \frac{d}{2} + i\boldsymbol{R}$ $\times$



$\vec{w} \in \boldsymbol{R}^d$

Amplitude Transforms

- It is useful to point out that the above transforms can be applied directly to the S-matrix elements.

Massive scalar

$$\widetilde{\mathcal{A}}(\Delta_i, \vec{w}_i'(\vec{w}_i)) = \prod_{k=1}^{n} \left| \frac{\partial \vec{w}_k'}{\partial \vec{w}_k} \right|^{-\Delta_k/d} \widetilde{\mathcal{A}}(\Delta_i, \vec{w}_i)$$

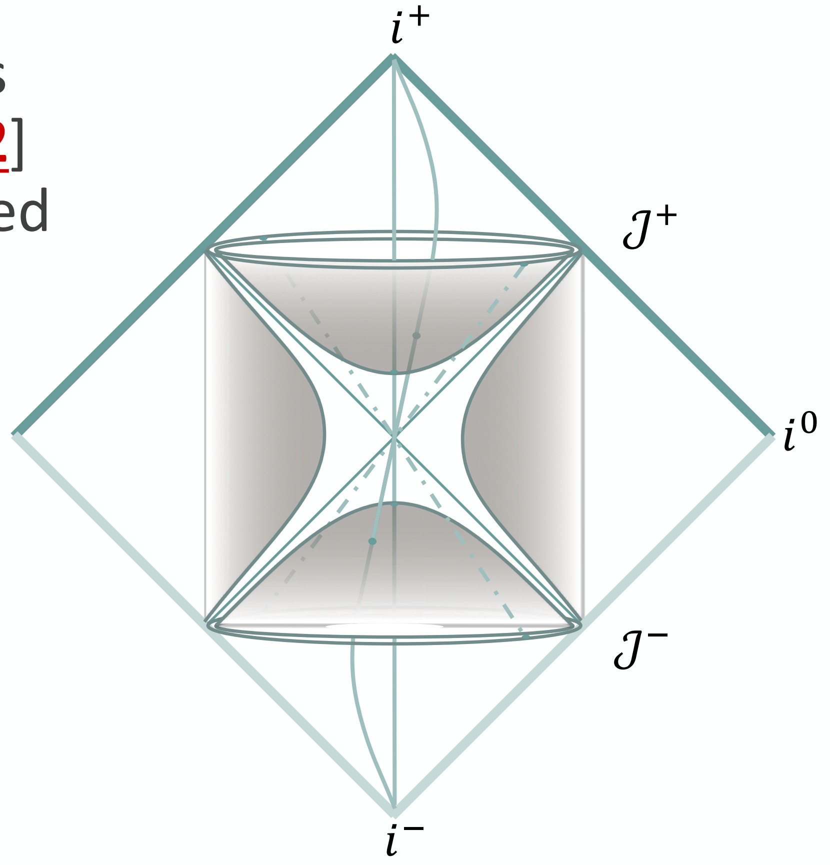
$$\widetilde{\mathcal{A}}(\Delta_i, \vec{w}_i) \equiv \prod_{k=1}^{n} \int_{H_{d+1}} [d\hat{p}_k]\, G_{\Delta_k}(\hat{p}_k; \vec{w}_k)\ \ \mathcal{A}(\pm m_i \hat{p}_i^{\mu})$$

$m = 0$

$$\widetilde{\mathcal{A}}(\Delta_i, \vec{w}_i) \equiv \prod_{k=1}^{n} \int_0^{\infty} d\omega_k \omega_k^{\Delta-1}\, \mathcal{A}(\pm \omega_k q_k^{\mu})$$

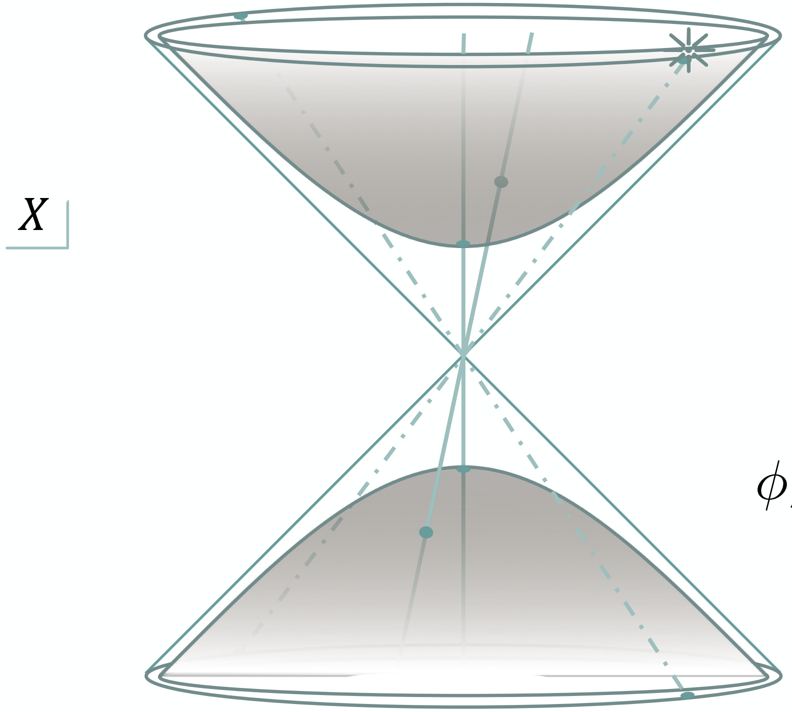
Amplitude Transforms

- Note that transforming momentum space amplitudes directly, is an alternative to previous approaches [**hep-th/0303006**,**arXiv:1609.00732**] towards flat space holography, which have looked at a foliation of Minkowski space to reproduce AdS/CFT, dS/CFT on each slice.



Example: Massive Scalar 3pt

- For d=2, we use the projective coordinate w, for the celestial sphere CS^2 at the boundary of the lightcone from the origin. w undergoes mobius transformations when the spacetime undergoes Lorentz transformations



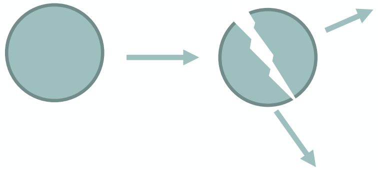
$$w = \frac{X^1 + iX^2}{X^0 + X^3} \qquad w \to \frac{aw+b}{cw+d}$$

- The highest weight states now look like quasi-primaries under SL(2,C)

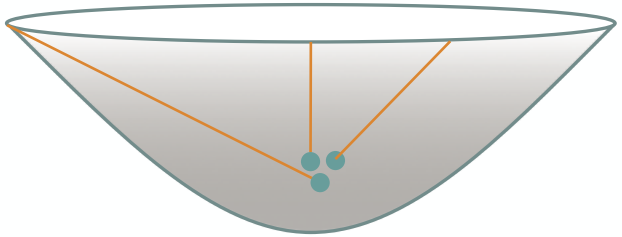
$$\phi_{\Delta,m}\left(\Lambda^{\mu}{}_{\nu}X^{\nu}; \frac{aw+b}{cw+d}, \frac{\bar{a}\bar{w}+\bar{b}}{\bar{c}\bar{w}+\bar{d}}\right) = |cw+d|^{2\Delta}\phi_{\Delta,m}\left(X^{\mu}; w, \bar{w}\right)$$

Example: Massive Scalar 3pt

- Lorentz covariance is built into the definition of the basis. If non-zero/finite 4D Lorentz covariance dictates 2D-correlator form.
- The behavior of low-point "correlation functions" is strongly dictated by momentum conservation in the bulk. Special scattering configurations can be used to get Witten diagram-like results.



$$2(1+\epsilon)m \to m+m$$



$$\tilde{\mathcal{A}}(w_i,\bar{w}_i) = \frac{i2^{\frac{9}{2}}\pi^6\lambda\Gamma(\frac{\Delta_1+\Delta_2+\Delta_3-2}{2})\Gamma(\frac{\Delta_1+\Delta_2-\Delta_3}{2})\Gamma(\frac{\Delta_1-\Delta_2+\Delta_3}{2})\Gamma(\frac{-\Delta_1+\Delta_2+\Delta_3}{2})\sqrt{\epsilon}}{m^4\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)|w_1-w_2|^{\Delta_1+\Delta_2-\Delta_3}|w_2-w_3|^{\Delta_2+\Delta_3-\Delta_1}|w_3-w_1|^{\Delta_3+\Delta_1-\Delta_2}} + \mathcal{O}(\epsilon)$$

[arXiv:1701.00049]

Example: MHV Mellin

- Momentum conservation strongly dictates the form of low point Mellin amplitudes. If we think of correlation functions of Mellin operators, we see the contact nature of the two point function already from the scalar Mellin modes:

$$\mathfrak{a}_\lambda(\hat{q}) \equiv \int_0^\infty d\omega\ \omega^{i\lambda} a(\omega, \hat{q}) \qquad \langle 0|\mathfrak{a}_{\lambda'}(\hat{q}')\mathfrak{a}^\dagger_\lambda(\hat{q})|0\rangle = (2\pi)^4\delta(\lambda-\lambda')\delta^{(2)}(w_1-w_2)$$

- For MHV amplitudes (and any theory with scale invariance) one finds that the Mellin transformed amplitudes have a conservation-of-weight

$$\tilde{\mathcal{A}}_{\Delta_1,\cdots,\Delta_n}(w_i,\bar{w}_i) \equiv \prod_{k=1}^n \int_0^\infty d\omega_k \omega_k^{i\lambda_k}\ \mathcal{A}(\omega_k q_k^\mu) \longrightarrow \tilde{A}_n = \prod_{k=1}^n \int_0^\infty d\omega_k \omega_k^{i\lambda_k} \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle \ldots \langle n1\rangle}\delta^4(\sum_k p_k)$$

$$\langle ij\rangle = 2\sqrt{\omega_i\omega_j}(w_i - w_j) \qquad \tilde{A} \propto \int_0^\infty ds s^{i\sum\lambda_k - 1} = 2\pi\delta(\sum \lambda_k)$$

Example: MHV Mellin

- Once you tell me the directions of scattering, the frequencies in the mellin integral get fixed, ie the momentum conserving delta functions localize the frequency integrals (and then some). For a $2 \to 2$ process with helicities $(- - + +)$

$$\tilde{\mathcal{A}}_4 = (-1)^{1+i\lambda_2+i\lambda_3} \frac{\pi}{2} \left[\frac{\eta^5}{1-\eta}\right]^{1/3} \delta(\mathrm{Im}[\eta])$$

$$\times\, \delta(\lambda_1+\lambda_2+\lambda_3+\lambda_4) \prod_{i<j}^{4} z_{ij}^{h/3-h_i-h_j} \bar{z}_{ij}^{\bar{h}/3-\bar{h}_i-\bar{h}_j}$$

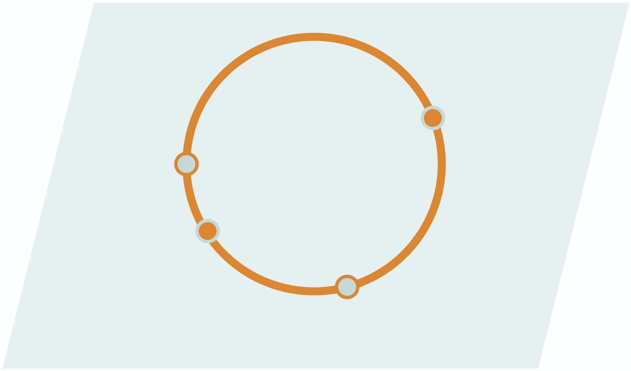
$$h^- = \frac{i}{2}\lambda \qquad \bar{h}^- = 1 + \frac{i}{2}\lambda$$

$$h^+ = 1 + \frac{i}{2}\lambda \qquad \bar{h}^+ = \frac{i}{2}\lambda$$

Example: MHV Mellin

- On-shell + momentum conserving kinematics restrict 2 → 2 reference directions to lie on a circle within the celestial sphere
- MHV 3pt has no support in (1,3) signature but can analytically continue to (2,2) signature with independent real coordinates

$$\tilde{\mathcal{A}}_3(\lambda_i; z_i, \bar{z}_i) = \pi(-1)^{i\lambda_1}\mathrm{sgn}(z_{23})\mathrm{sgn}(z_{13})\,\delta(\sum_i \lambda_i)\frac{\delta(\bar{z}_{13})\delta(\bar{z}_{12})}{z_{12}^{-1-i\lambda_3}\,z_{23}^{1-i\lambda_1}\,z_{13}^{1-i\lambda_2}}$$



What Needs To Be Done

- The current map interpreting S-matrix elements as 2D CFT correlators seems to imply either an exotic CFT 2 or that the map needs to be finessed… **Options?**

  - *Is there a better shadow-related basis?*

$$\mathcal{O}^{+}_{i\lambda}(w,\bar{w}) = \phi^{+}_{i\lambda}(w,\bar{w}) + C_{+,\lambda}\int d^2z \frac{1}{(z-w)^{2+i\lambda}(\bar{z}-\bar{w})^{i\lambda}}\phi^{-}_{-i\lambda}(z,\bar{z})$$
$$\mathcal{O}^{-}_{i\lambda}(w,\bar{w}) = \phi^{-}_{i\lambda}(w,\bar{w}) + C_{-,\lambda}\int d^2z \frac{1}{(z-w)^{i\lambda}(\bar{z}-\bar{w})^{2+i\lambda}}\phi^{+}_{-i\lambda}(z,\bar{z})$$

  - The mode combination that decouples in the soft limit is precisely a linear combination of Mellin and Mellin+shadow in the limit where $\operatorname{Im}\Delta = 0$:

$$\mathbf{a}_{-} \equiv a_{-}(\omega\hat{x}) - \frac{1}{2\pi}\int d^2w \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}} a_{+}(\omega\hat{y})$$

Understand the conformally soft limit!

What Needs To Be Done

- Illustrate some version of an OPE:

  - *Collinear limits in higher point amplitudes avoiding low point kinematic issues (Taylor)?*

  - *Relate worldsheet CFT to celestial sphere CFT*

  - *Consider full S-matrix rather than transfer matrix so that have nonzero 4pt functions even for free theory (w/ SH Shao?), then use recent literature on CFT principal series completeness relations to interpret intermediate exchanges.*

What Has Been Done

- Beautiful expressions for full mellin transform (which inherently probe UV structure) of string amplitudes [**arXiv:1806.05688**]
- Systematic n-pt N^kMHV [**arXiv:1711.08435**]
- 3D example of CB decomposition [**arXiv:1711.06138**]
- Interesting statments about symplectic pairing of conformally soft modes [**arXiv:1810.05219**]

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1810.05219

$$A^{\Delta,\pm}_{\mu;a}(X^\mu; w, \bar{w}) = \frac{\partial_a q_\mu}{(-q\cdot X \mp i\varepsilon)^\Delta} + \frac{(\partial_a q\cdot X)\, q_\mu}{(-q\cdot X \mp i\varepsilon)^{\Delta+1}}$$

$$A^{\Delta,\pm}_{\mu;a} = \frac{\Delta-1}{\Delta(\mp i)^\Delta \Gamma(\Delta)} V^{\Delta,\pm}_{\mu;a} + \partial_\mu \alpha^{\Delta,\pm}_a \qquad \alpha^{\Delta,\pm}_a(X^\mu; w, \bar{w}) = \frac{\partial_a q\cdot X}{\Delta(-q\cdot X \mp i\varepsilon)^\Delta}$$

$$V^{\Delta,\pm}_{\mu;a}(X^\mu; w, \bar{w}) = \partial_a q_\mu \int_0^\infty d\omega\, \omega^{\Delta-1} e^{\pm i\omega q\cdot X - \varepsilon\omega}$$

- The conformally soft limit is where $\Delta \to 1$. There the CB wavefunction is pure gauge. **arXiv:1609.00732**

1810.05219

- Donnay et al. identify a logarithmic mode as its symplectic partner

$$A^{\mathrm{G}}_{\mu;a} \equiv \lim_{\Delta\to 1} A^{\Delta,\pm}_{\mu;a} = \partial_\mu \alpha^1_a \qquad \alpha^1_a = -\frac{\partial_a q\cdot X}{q\cdot X}$$

$$A^{\log,\pm}_{\mu;a} \equiv \lim_{\Delta\to 1} \partial_\Delta \left(A^{\Delta,\pm}_{\mu;a} + \tilde{A}^{2-\Delta,\pm}_{\mu;a} \right) \qquad A^{\log,\pm}_{\mu;a} = -\log\left[-X^2 \mp 2i\varepsilon X^0 - \varepsilon^2\right] \partial_\mu \left(\frac{\partial_a (q\cdot X \pm i\varepsilon q^0)}{-q\cdot X \mp i\varepsilon q^0} \right)$$

$$F^{\mathrm{CS}}_{\mu\nu;a} \equiv \frac{1}{2\pi i} \left(F^{\log,+}_{\mu\nu;a} - F^{\log,-}_{\mu\nu;a} \right) \qquad A^{\mathrm{CS}}_{\mu;a} = (q\cdot X)\log[X^2] A^{\mathrm{G}}_{\mu;a} \delta(q\cdot X) + A^{\mathrm{G}}_{\mu;a} \Theta\left(X^2\right)$$

1810.05219

- Donnay et al. identify a logarithmic mode as its symplectic partner

$$\hat{A}_\mu(X) = \int \frac{d^2w\, d\lambda\sqrt{2}}{(2\pi)^4} \left(a_{\lambda w} V^{*1-i\lambda,+}_{\mu;\bar{w}}(X) + a^\dagger_{\lambda\bar{w}} V^{1+i\lambda,-}_{\mu;w}(X) + (w \leftrightarrow \bar{w}) \right)$$
$$+ \int \frac{d^2w}{8\pi^2} \left(S_w A^{\mathrm{G}}_{\mu;w}(X) + J_w A^{\mathrm{CS}}_{\mu;w}(X) + (w \leftrightarrow \bar{w}) \right)$$

$$[a_{\lambda w}(w), a^\dagger_{\lambda' w'}(w')] = \frac{1}{2}(2\pi)^4 \delta(\lambda - \lambda')\delta^{(2)}(w - w')$$

$$[J_w(w), S_{w'}(w')] = 8\pi^2 \delta^{(2)}(w - w')$$

Final Thoughts

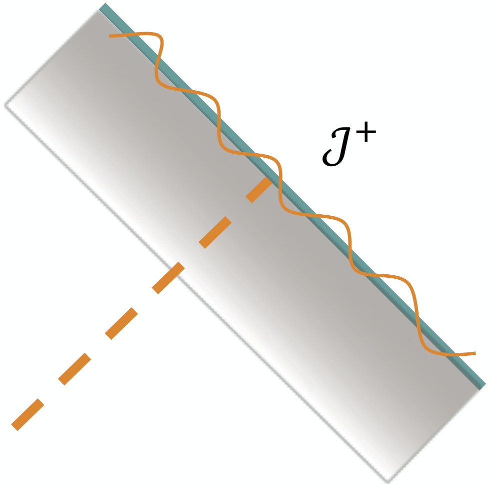
- This conformally soft phase space analysis / augmentation is the analog of what started this story at null infinity.
- In addition to learning more about the IR description as a side effect of running into issues with contact-term two point functions (with shadows as a way out)…
- …the completeness of the conformal basis is needed to justify an effective `fixed- Δ hyperbolic slicing’ in current attempts to connect more with the standard AdS3/CFT2 dictionary.

Final Thoughts

$$h_{uu} = \sum_{n=2}^{\infty} \frac{h_{uu}^{(n)}}{r^n} + \sum_{n=1}^{\infty} \frac{\tilde{h}_{uu}^{(n)} \log r}{r^n}, \qquad h_{ur} = \sum_{n=2}^{\infty} \frac{h_{ur}^{(n)}}{r^n} + \sum_{n=2}^{\infty} \frac{\tilde{h}_{ur}^{(n)} \log r}{r^n},$$

$$h_{rr} = \sum_{n=3}^{\infty} \frac{h_{rr}^{(n)}}{r^n} + \sum_{n=3}^{\infty} \frac{\tilde{h}_{rr}^{(n)} \log r}{r^n}, \qquad h_{uA} = \sum_{n=1}^{\infty} \frac{h_{uA}^{(n)}}{r^n} + \sum_{n=1}^{\infty} \frac{\tilde{h}_{uA}^{(n)} \log r}{r^n},$$

$$h_{rA} = \sum_{n=1}^{\infty} \frac{h_{rA}^{(n)}}{r^n} + \sum_{n=2}^{\infty} \frac{\tilde{h}_{rA}^{(n)} \log r}{r^n}, \quad h_{AB} = h_{AB}^{(-1)} r + \sum_{n=0}^{\infty} \frac{h_{AB}^{(n)}}{r^n} + \sum_{n=0}^{\infty} \frac{\tilde{h}_{AB}^{(n)} \log r}{r^n}.$$



$$\begin{aligned}
[\Box h_{uu}]^{(n)} &= 2(n-2)\partial_u h_{uu}^{(n-1)} + [D^2 + (n-2)(n-3)]h_{uu}^{(n-2)} \\
&\quad + (-2n+5)\tilde{h}_{uu}^{(n-2)} - 2\partial_u \tilde{h}_{uu}^{(n-1)} \\
[\Box h_{ur}]^{(n)} &= 2(n-2)\partial_u h_{ur}^{(n-1)} + [D^2 + (n-2)(n-3) - 2]h_{ur}^{(n-2)} + 2h_{uu}^{(n-2)} - 2D^A h_{uA}^{(n-3)} \\
&\quad + (-2n+5)\tilde{h}_{ur}^{(n-2)} - 2\partial_u \tilde{h}_{ur}^{(n-1)} \\
[\Box h_{rr}]^{(n)} &= 2(n-2)\partial_u h_{rr}^{(n-1)} + [D^2 + (n-2)(n-3)]h_{rr}^{(n-2)} \\
&\quad + 4\left(h_{ru}^{(n-2)} - h_{rr}^{(n-2)}\right) - 4D^A h_{rA}^{(n-3)} + 2\gamma^{AB} h_{AB}^{(n-4)} \\
&\quad + (-2n+5)\tilde{h}_{rr}^{(n-2)} - 2\partial_u \tilde{h}_{rr}^{(n-1)} \\
[\Box h_{uA}]^{(n)} &= 2(n-1)\partial_u h_{uA}^{(n-1)} + [D^2 + (n-2)(n-1) - 1]h_{uA}^{(n-2)} - 2\partial_A \left(h_{uu}^{(n-1)} - h_{ur}^{(n-1)}\right) \\
&\quad + (-2n+3)\tilde{h}_{uA}^{(n-2)} - 2\partial_u \tilde{h}_{uA}^{(n-1)} \\
[\Box h_{rA}]^{(n)} &= 2(n-1)\partial_u h_{rA}^{(n-1)} + [D^2 + (n-2)(n-1) - 5]h_{rA}^{(n-2)} - 2\partial_A \left(h_{ur}^{(n-1)} - h_{rr}^{(n-1)}\right) \\
&\quad - 2D^B h_{AB}^{(n-3)} + 4h_{uA}^{(n-2)} \\
&\quad + (-2n+3)\tilde{h}_{rA}^{(n-2)} - 2\partial_u \tilde{h}_{rA}^{(n-1)} \\
[\Box h_{AB}]^{(n)} &= 2n\partial_u h_{AB}^{(n-1)} + [D^2 + (n-2)(n+1)]h_{AB}^{(n-2)} \\
&\quad - 2\left(D_A h_{uB}^{(n-1)} - D_A h_{rB}^{(n-1)} + D_B h_{uA}^{(n-1)} - D_B h_{rA}^{(n-1)}\right) + 2\gamma_{AB}\left(h_{uu}^{(n)} - 2h_{ur}^{(n)} + h_{rr}^{(n)}\right) \\
&\quad + (-2n+5)\tilde{h}_{AB}^{(n-2)} + 2\partial_u \tilde{h}_{AB}^{(n-1)}
\end{aligned}$$

Final Thoughts

- The conformal basis solutions are homogeneous under τ ➞ λ τ, with dilation weight corresponding to their principal series weight

$$\nabla^\mu F_{\mu\nu} = e^2 j_\nu$$

$$\begin{aligned}
\tau^3\nabla^\mu F_{\mu\tau} &= ((\rho\partial_\rho)^2 + 2(\rho\partial_\rho) + \tfrac{4}{\rho^2}\partial_z\partial_{\bar z})(\tau A_\tau) - ((\rho\partial_\rho)+2)(\tau\partial_\tau)(\rho A_\rho) - \tfrac{2}{\rho^2}(\tau\partial_\tau)(\partial_{\bar z}A_z + \partial_z A_{\bar z}) \\
\rho\tau^2\nabla^\mu F_{\mu\rho} &= ((\tau\partial_\tau)+4)(\rho\partial_\rho)(\tau A_\tau) - ((\tau\partial_\tau)^2 + 4(\tau\partial_\tau) + \tfrac{4}{\rho^2}\partial_z\partial_{\bar z})(\rho A_\rho) - \tfrac{2}{\rho^2}(\rho\partial_\rho)(\partial_{\bar z}A_z + \partial_z A_{\bar z}) \\
\tau^2\nabla^\mu F_{\mu z} &= ((\tau\partial_\tau)+4)\partial_z(\tau A_\tau) - ((\rho\partial_\rho)+4)\partial_z(\rho A_\rho) \\
&\quad + (-(\tau\partial_\tau)^2 + (\tau\partial_\tau) + (\rho\partial_\rho)^2 + 4(\rho\partial_\rho) + \tfrac{2}{\rho^2}\partial_z\partial_{\bar z})A_z - \tfrac{2}{\rho^2}\partial_z^2 A_{\bar z}
\end{aligned}$$

$$\begin{aligned}
x^0 &= \frac{1}{2}(\frac{\tau}{\rho} + \tau\rho(1+z\bar z)) \\
x^1 &= \frac{1}{2}\tau\rho(z+\bar z) \\
x^2 &= -\frac{i}{2}\tau\rho(z-\bar z) \\
x^3 &= \frac{1}{2}(\frac{\tau}{\rho} - \tau\rho(1-z\bar z))
\end{aligned}$$

$$\begin{aligned}
u &\mapsto \frac{\tau}{\rho} \\
r &\mapsto \tau\rho \\
z &\mapsto z \\
\bar z &\mapsto \bar z
\end{aligned}$$