4pt in 2+1

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This note considers the four point function of four massless fields in a 2+1 dimensional Massive-massless-massless cubic theory. I show that much of the work in 3+1 carries over straightforwardly (as if restricting to the celestial circle as far as the kinematics is concerned). The feature allowing a conformal block decomposition is that the four point function in 3 dimensions in a non-scale invariant theory does not over-constrain the Mellin transform. Unitarity in 2+1 follows from the change of basis for a theory that is unitary in momentum space, while the conformal block decomposition should be used to reveal features of the dual picture.

I. FROM 3+1 TO 2+1

In 2+1 dimensions a null momentum can be parameterized by

\[ p^\mu = \omega (1 + z^2, 2z, 1 - z^2) \]  
(I.1)

where \( z \in \mathbb{R} \). For four massless external momenta, the delta function coming from momentum conservation has the form:

\[ \delta^{(3)}(p_1 + p_2 - p_3 - p_4) = \frac{1}{4|z_{12}z_{34}|} \delta(\omega_1 + \omega_2, \frac{z_{23}z_{24}}{z_{12}z_{34}}) \times \delta(\omega_3 - \omega_4, \frac{z_{23}Z_{24}}{z_{12}z_{34}}). \]  
(I.2)

We get some nice intuition for the above result when we consider setting \( z_i = \bar{z}_i \) in the 3+1 four point Mellin paper, and removing the delta function for the real cross ratios since we are in one dimension less and on a locus where what would be the reality constraint of the cross ratios is already satisfied. Note that aside from the measure, the values of \( \omega_i \) can be directly compared to 4.2 in the Mellin paper and agree on the \( z_i = \bar{z}_i \) locus. Note also that the 12 ↔ 34 channel corresponds to \( x = \frac{z_{12}z_{34}}{z_{13}z_{24}} > 1 \) as can be seen by demanding the localized \( \omega_i \) be positive and noting that for instance \( \text{sgn}(\frac{z_{23}z_{24}}{z_{12}z_{34}}) = \text{sgn}(\frac{z_{23}z_{24}}{z_{13}z_{24}}) = \text{sgn}(x) \).

For a theory that is not scale invariant, one expects that the Mellin transform of a 4pt function will localize 3 of the frequency integrals. Having a massive intermediate propagator in a three point theory then gives hope for the final integral to be finite.

If I am considering charged massless scalars, then a \( g\phi\rho^\ast\varphi \) interaction can be set up to have only \( s \) and \( t \) channels. The desired amplitude is proportional to

\[ \mathcal{A} = g^2 \left[ \frac{1}{(p_1 + p_2)^2 + M^2} + \frac{1}{(p_1 - p_3)^2 + M^2} \right] \delta^{(3)}(p_1 + p_2 - p_3 - p_4) \]  
(I.3)

Then

\[ \tilde{\mathcal{A}} = \prod_{i=1}^{4} \int dw_i \omega_i^{-\frac{1}{2} + i\lambda_i} \mathcal{A} \]

\[ = - \frac{2^{-2 - 4\Sigma \lambda}}{M^2 - 2 \Sigma \lambda} \text{sech}(\frac{\pi}{2} \Sigma \lambda) \theta(\frac{z_{12}z_{24}}{z_{13}z_{24}} - 1) \times \left( \frac{z_{23}z_{24}}{z_{13}z_{24}} \right)^{\frac{1}{2} + \Sigma \lambda} \left[ \frac{z_{23}z_{24}}{z_{13}z_{24}}^{\frac{1}{2} + \Sigma \lambda} + \left( \frac{z_{23}z_{24}}{z_{13}z_{24}} \right)^{-\frac{1}{2} - \Sigma \lambda} \right] \]

\[ \times \left( - \frac{z_{23}z_{24}}{z_{13}z_{24}} \right)^{-\frac{1}{2} + \Sigma \lambda} \left( \frac{z_{23}z_{24}}{z_{13}z_{24}}^{\frac{1}{2} + \Sigma \lambda} - \left( \frac{z_{23}z_{24}}{z_{13}z_{24}} \right)^{-\frac{1}{2} - \Sigma \lambda} \right) \]  
(I.4)

The support of the delta function is suppressed in the integrations and guarantees the localized values of \( \omega_i \) are positive. Now the point is that this answer has non-singular support for the \( z_i \) so that the 1 dimensional analog of Mack et al.’s decomposition theorem would say that we can write this in terms of conformal blocks. Meanwhile any statement of unitarity in 2+1 is guaranteed by the change of basis being consistent and starting with a momentum space \( S \)-matrix that is already unitary. One should look at the conformal block decomposition for statements from the dual perspective.

II. CONFORMAL BLOCK DECOMPOSITION

We will now use some equations from Jackiw & Pi 1205.0443. First their (2.4) gives the general form of a four point function

\[ (\mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\mathcal{O}_3(z_3)\mathcal{O}_4(z_4)) = \frac{F(x)}{z_{12}z_{34}z_{13}z_{24}} \]  
(II.1)

where \( \Delta_{ij} = \Delta_i - \Delta_j, x = \frac{z_{12}z_{34}}{z_{13}z_{24}} \) and \( F(x) \) is a sum over blocks (2.5)

\[ F(x) = \sum_i b_i B_i(x) \]  
(II.2)

where (2.12)

\[ B = \frac{x^\Delta - 1}{2} F_1(\Delta - \Delta_{12}, \Delta + \Delta_{34}; 2\Delta; x) \]  
(II.3)

For our scalars \( \Delta_i = \frac{1}{2} + i\lambda_i \). We thus want to rewrite our (II.4) in terms of a sum/integral over \( \Delta_i \). Factorizing out the covariant piece a la (II.1), we find

\[ F(x) = \frac{2^{-2 - 4\Sigma \lambda}}{M^2 - 2 \Sigma \lambda} \text{sech}(\frac{\pi}{2} \Sigma \lambda) \theta(-1)^{\frac{1}{2} + \frac{1}{2}} \left[ 1 + \left( \frac{1}{1 - x} \right)^{\frac{1}{2} + \Sigma \lambda} \right] \]  
(II.4)

\[ \theta(x - 1)x \left[ \frac{1}{1 - x} \right]^{\frac{1}{2} + \frac{1}{2}} \left[ 1 + \left( \frac{1}{1 - x} \right)^{\frac{1}{2} + \Sigma \lambda} \right] \]  
(II.5)

Keeping in mind the crossing vs. range of \( x \) and avoiding the phase and \( z_i \) dependent sign for the moment, we will now construct the conformal block decomposition of the simpler function:

\[ f(x) = \frac{1}{1 - x} \left[ \frac{1}{1 - x} \right]^{\frac{1}{2} + \frac{1}{2}} \left[ 1 + \left( \frac{1}{1 - x} \right)^{\frac{1}{2} + \Sigma \lambda} \right] \]  
(II.5)
What then remains at this point is to find a principal series (or otherwise) decomposition of functions of the form \(x^a(1-x)^{-\frac{1}{2}(1-\Delta_{12}+\Delta_{34})}\) a la (II.2), much like the author and S.H. Shao had attempted in the higher dimensional case, but now with the first hurdle of removing any singularity via shadows taken care of.