1. In this problem, we will study the gravitational memory effect experienced by inertial observers due to supertranslations.

(a) Working in retarded Bondi coordinates, determine the 4-velocity of an inertial time-like observer (i.e. an observer travelling along a time-like geodesic) in the large-$r$ limit. Determine the $u$ and $r$ components to order $\frac{1}{r}$ and the $\bar{z}$ and $\bar{z}$ components to order $\frac{1}{r^2}$. Normalize the 4-velocity in the usual way $(v^\mu v_\mu = -1)$, and require that at $u = u_0$ the trajectory pass through the point $(r_0, z_0, \bar{z}_0)$ where $r_0 \gg u_0$.

(b) Now consider a nearby inertial observer who passes through the $(r_1, z_1, \bar{z}_1)$ at $u = u_0$, where $r_1 = r_0 + \delta r$ with $\delta r \ll r_0$. Determine the proper distance between the observers to leading order in $\delta r$.

(c) A burst of radiation of the form

$$T^M_{uu}(u, r, z, \bar{z}) = \frac{\mu}{4\pi r^2} \delta(u - u_{\text{rad}})\delta^2(z - z_{\text{rad}}), \quad u_{\text{rad}} > u_0, \quad \frac{|u_{\text{rad}} - u_0|}{r_0} \ll 1 , \quad (1)$$

passes the observers. Determine the change in proper distance between the observers as a result of the passage of this radiation.

(d) Find the supertranslation $f$ that gives rise to the same change in proper distance.
2. Consider the Vaidya black hole created by a spherically symmetric null shockwave at $v = 0$. We define the horizon large gauge charge as

$$Q^H_\varepsilon \equiv \frac{1}{e^2} \int_{H^+} \varepsilon (*F) .$$

(2)

where $H^+$ is the future of the horizon of the Vaidya black hole.

(a) Using the constraints derived from Maxwell’s equations on $H$, rewrite $Q^H_\varepsilon$ as an integral over $H$ and determine the soft and hard part of this charge. Write the soft charge explicitly in the $(v, r, z, \bar{z})$ coordinates.

(b) Determine the symplectic form for the gauge fields on $H$. Hint: You can use the symplectic form $\Omega_\Sigma$ that was derived in Problem Set 1 and set $\Sigma = H$. Write it out explicitly in the $(v, r, z, \bar{z})$ coordinates.

(c) By using the explicit forms of the charge and symplectic form derived in parts (a) and (b), show that $Q^H_\varepsilon$ generates large gauge transformations on the horizon, i.e.

$$[Q^H_\varepsilon, A_z]_{|_{H^+}} = i \partial_z \varepsilon .$$

(3)

3. Consider a Schwarzschild black hole with mass $M$ perturbed by a null shockwave at $v = v_0$ described by the stress tensor

$$T_{vv} = \frac{1}{4\pi r^2} \left[ \mu + \frac{1}{4} D^2 \left( D^2 + 2 \right) f \right] \delta (v - v_0) - \frac{3M}{8\pi r^3} D^2 f \delta (v - v_0) ,$$

$$T_{vA} = -\frac{3M}{8\pi r^2} D_A f \delta (v - v_0) .$$

(4)

(a) Solve the linearized Einstein’s equations about the initial Schwarzschild solution in Bondi gauge. Assume that only the zero mode of the Bondi mass aspect on $I^-$ changes under this perturbation.

(b) Show that the final black hole (after $v > v_0$) can be obtained instead by sending a spherically symmetric null shockwave with stress tensor

$$T_{vv} = \frac{\mu}{4\pi r^2} \delta (v - v_0)$$

(5)

followed by an infinitesimal supertranslation.