Covariant Derivatives and the Hamilton-Jacobi Equation

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I define a covariant derivative to simplify how higher order derivatives act on a classical generating function.

When studying the connection between classical and quantum mechanics, it would be nice to have a differential operator which, when acting repeatedly on some function $e^{\frac{2iHx}{\hbar}}$ pulls down powers of the derivatives of the function within the exponent.

Consider the results of “Wavefunctions and the Hamilton-Jacobi Equation.” There, I performed a canonical change of variables from $(q_i, p_i)$ to constants $(Q_i, P_i)$:

$$ p_i \dot{q}_i - H(q, p, t) = P_i \dot{Q}_i - K(Q, P, t) + \frac{dF}{dt} $$

(1)

where $F = S(q, P, t) - P_i Q_i$, and found:

$$ H = -\frac{\partial S}{\partial t} p_i = \frac{\partial S}{\partial \dot{q}_i} Q_i = \frac{\partial S}{\partial \dot{P}_i} P_i. $$

(2)

when $K = 0$. At first order, the function $e^{\frac{2iS}{\hbar}}$ had the property that ordinary multiplication by the value of $H$ or $p_j$ was equivalent to acting with a differential operator:

$$ H \cdot e^{\frac{2iS}{\hbar}} = -\frac{\partial S}{\partial t} e^{\frac{2iS}{\hbar}} = i\hbar \partial_t e^{\frac{2iS}{\hbar}} $$

$$ p_j \cdot e^{\frac{2iS}{\hbar}} = \frac{\partial S}{\partial \dot{q}_j} e^{\frac{2iS}{\hbar}} = \hbar \partial_{\dot{q}_j} e^{\frac{2iS}{\hbar}}. $$

(3)

Moreover, this connection between multiplication and a differential operator held at first order for arbitrary superpositions:

$$ \Phi(q, t) = \int \mathbb{P}(P_i) e^{\frac{2iS(q, p, t)}{\hbar}} dP_i. $$

(4)

In this paper, I will consider the one dimensional case $q = x$ and define a covariant derivative such that acting on $e^{\frac{2iS(x, t)}{\hbar}}$ a total of $n$ times with the operator $\frac{\hbar}{i} \nabla_x$ is exactly equivalent to multiplying $e^{\frac{2iS(x, t)}{\hbar}}$ by $p^n = (\frac{\partial S}{\partial q})^n$.

The key is to treat $e^{\frac{2iS(x, t)}{\hbar}}$ as a scalar, with a non-trivial one-dimensional spatial metric $g_{xx} = (\frac{\partial S}{\partial q})^2$. Then there is a non-zero connection $\Gamma^x_{xx} = \frac{1}{2} g^{xx} \partial_x g_{xx} = \frac{S''}{s''}$, where primes denote partial derivatives with respect to $x$.

If I treat $\hat{p}^n e^{\frac{2iS(x, t)}{\hbar}} \equiv (\frac{\hbar}{i})^n \nabla_x e^{\frac{2iS(x, t)}{\hbar}}$ as a covariant rank-$n$ tensor, I find:

$$ \nabla_x e^{\frac{2iS}{\hbar}} = \nabla_x e^{\frac{2iS}{\hbar}} - \frac{\partial S}{\partial x} e^{\frac{2iS}{\hbar}}. $$

(5)

It is quick to check for $n = 1$ that $\nabla_1 e^{\frac{2iS}{\hbar}} = \nabla_x e^{\frac{2iS}{\hbar}} - \frac{\partial S}{\partial x} e^{\frac{2iS}{\hbar}}.$ If it is true that $\nabla_{n-1} e^{\frac{2iS}{\hbar}} = (\nabla_S)^{n-1} e^{\frac{2iS}{\hbar}}$, then:

$$ \nabla_n e^{\frac{2iS}{\hbar}} = \partial_x (\frac{2iS}{\hbar})^{n-1} e^{\frac{2iS}{\hbar}} = (n-1) (\frac{S''}{s''})^{n-1} e^{\frac{2iS}{\hbar}} = (n-1) (\frac{S''}{s''})(\frac{S''}{s''})^{n-2} e^{\frac{2iS}{\hbar}} = (\frac{S''}{s''}) e^{\frac{2iS}{\hbar}}. $$

(6)

since $\Gamma^x_{xx} = \frac{S''}{s''}$, so $(\frac{\hbar}{i})^n e^{\frac{2iS(x, t)}{\hbar}} = \hat{p}^n e^{\frac{2iS(x, t)}{\hbar}} = (\frac{\hbar}{i})^n \nabla_x e^{\frac{2iS(x, t)}{\hbar}}$ holds by induction.

If I define a partition function expectation value:

$$ \langle \mathcal{O}(x, p) \rangle = \int \mathbb{P}(P) dP dq \mathcal{O}(x, p) e^{\frac{2iS(x, t)}{\hbar}} $$

(7)

then this is equivalent to:

$$ \langle \mathcal{O}(x, p) \rangle = \int \mathbb{P}(P) dP dx : \mathcal{O}(x, \hat{p}) : e^{\frac{2iS(x, t)}{\hbar}} $$

(8)

where the normal ordered operator is defined such that all of the momentum operators appear on the right. The direct correspondence between $x^n p^m = x^n (S')^m$ in $\mathcal{O}$ and $x^n \hat{p}^m = x^n (\frac{\hbar}{i})^m \nabla_m$ in $: \mathcal{O}(x, \hat{p}) :$ thus follows from the composition property of the covariant derivative.

Summarizing Equation 8 as $\langle \mathcal{O}(q, p) \rangle = : \mathcal{O}(q, \hat{p}) :$, one finds that for an operator which does not explicitly depend on time:

$$ \langle \mathcal{O}(x, \hat{p}) \rangle = \langle \{ \mathcal{O}, H \} \rangle $$

(9)

versus

$$ \langle \mathcal{O}(x, \hat{p}) \rangle = \langle \{ \mathcal{O}, H \} \rangle = \frac{1}{\hbar} \langle [\mathcal{O}, : H :] \rangle $$

(10)

The last equality comes from considering a generic term in the series expansion of $\mathcal{O}(x, p)$

$$ \langle \{ x^n \hat{p}^m, x^r \hat{p}^s \} \rangle = \langle \{ x^n \hat{p}^m, x^r \hat{p}^s \} \rangle $$

(11)

where some care must be taken when specifying what it means to normal order the commutator (e.g. I would want to have $: [x, \hat{p}] := i\hbar$ and not $: [x, \hat{p}] := x\hat{p} - : \hat{p} x : = 0$).

Equation 9 is similar to Ehrenfest’s Theorem. There is a natural association between the Poisson Bracket of classical mechanics and the normal ordered commutator of normal ordered operators.