Solving the Shrödinger Equation Using a Complex Gauge

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I show a quick route to the $n = \ell + 1$ wave functions and energies for the Hydrogen atom using a method which is also applied to find the ground state of the simple harmonic oscillator.

In this paper, I modify the procedure normally used to change between gauges for a magnetic field, and employ it as a quick route to finding particular solutions to the Schrödinger Equation for 1/r and quadratic potentials. I will start the radial Schrödingier Equation for Hydrogen:

$$Hu_{\ell}(r) = \left[\frac{p^2}{2m} + \frac{\hbar^2 \ell(\ell+1)}{2mr^2} - \frac{e^2}{r}\right] u_{\ell}(r).$$
(1)

Here, $u_{\ell}(r)$ obeys an effective one-dimensional Hamiltonian with the restriction that $u_{\ell}(0) = 0$. A threedimensional solution to the full Schrödingier Equation is then:

$$\Psi = \frac{u_\ell}{r} Y_{\ell m}.$$
 (2)

In the presence of a magnetic field, the Hamiltonian for an electron would be modified by changing $p_i \rightarrow p_i + \frac{e}{c}A_i$ where \vec{A} is the vector potential: $\vec{\nabla} \times \vec{A} = \vec{B}$. Changing \vec{A} by a gradient $\vec{\nabla}\lambda$ does not modify \vec{B} but will add a phase to the wavefunction, such that the new solution is:

$$\psi' = \exp\left[\frac{-ie\lambda}{\hbar c}\right]\psi. \tag{3}$$

For motion along a single direction \hat{x} with no magnetic field, we can change the \hat{x} component of the conventional vector potential $\vec{A} = 0$ by an arbitrary function $\frac{c}{e}f(x)$. The new vector potential $\frac{c}{e}f(x)\hat{x}$ will still have zero curl and the solution to the Schrödingier Equation:

$$H\psi' = \left[\frac{(p-f)^2}{2m} + V\right]\psi' \tag{4}$$

will also describe the motion of an electron in a potential V with no magnetic field. Using Equation 3, the solution to the original Schrödingier Equation will be given by:

$$\psi = \exp\left[\frac{i}{\hbar} \int f dx\right] \psi'.$$
 (5)

The radial Schrödingier Equation for Hydrogen is an example where the introduction of a vector potential-like term can actually simplify finding u_{ℓ} for the ground state. Let:

$$\tilde{H}\tilde{u}_{\ell}(r) = \left[\frac{1}{2m}\left(p + \frac{k}{r}\right)^2 + \frac{\hbar^2\ell(\ell+1)}{2mr^2} - \frac{e^2}{r}\right]\tilde{u}_{\ell}(r) \quad (6)$$

where I use the notation \tilde{H} since I will not restrict k to being real. Although this makes \tilde{H} non-hermitian,

the equation for u_{ℓ} in terms of \tilde{u}_{ℓ} is still valid, and it is useful to think of it as a generalized change in gauge.

Expanding H and arranging p to the right of 1/r in the expansion of the squared term gives:

$$\tilde{H} = \frac{p^2}{2m} + \frac{k}{mr}p + \frac{k^2 + i\hbar k}{2mr^2} + \frac{\hbar^2\ell(\ell+1)}{2mr^2} - \frac{e^2}{r}.$$
 (7)

For $k = \{i\hbar\ell, -i\hbar(\ell+1)\}$ the $1/r^2$ terms in Equation 9 cancel. These choices for k would give:

$$u_{\ell} = \{ \tilde{u}_{\ell} \cdot r^{-\ell}, \tilde{u}_{\ell} \cdot r^{(\ell+1)} \}.$$
(8)

The restriction $u_{\ell}(0) = 0$ leads us to choose the second option: $k = -i\hbar(\ell+1)$. This gives a differential equation for the stationary states \tilde{u}_{ℓ} :

$$E\tilde{u}_{\ell} = \left[-\frac{\hbar^2}{2m}\partial_r^2 - \frac{\hbar^2(\ell+1)}{mr}\partial_r - \frac{e^2}{r}\right]\tilde{u}_{\ell}.$$
 (9)

for which it is seen that an exponential solution $e^{\alpha r}$ can be chosen such that the 1/r terms cancel: $\alpha = \frac{-me^2}{\hbar^2(\ell+1)}$.

I have thus found a solution with energy $E = \frac{-\hbar^2 \alpha^2}{2m} = \frac{-me^4}{2\hbar^2(\ell+1)^2}$. The corresponding wave functions for $\ell = 0, 1, 2...$ are:

$$\Psi = Ar^{\ell} \exp\left[\frac{-r}{(\ell+1)a_0}\right] Y_{\ell m} \tag{10}$$

for some normalization constant A and Bohr radius $a_0 = \frac{\hbar^2}{me^2}$. These correspond to the $n = \ell + 1$ states of the Hydrogen atom.

This derivation takes advantage of a special cancellation allowed for a 1/r potential that makes an exponential solution to the resulting differential equation apparent. This technique of introducing an effective gauge can also be applied to a quadratic potential:

$$\tilde{H} = \frac{1}{2m} (p + im\omega x)^2 + \frac{1}{2}m\omega^2 x^2$$

$$= \frac{p^2}{2m} + i\omega xp + \frac{\hbar\omega}{2}$$
(11)

has $\psi' = const.$ as a solution with energy $E = \frac{\hbar\omega}{2}$. Equation 5 gives a solution to the original Hamiltonian:

$$\psi = A \exp\left[\frac{i}{\hbar} \int im\omega x dx\right] = A \exp\left[-\frac{m\omega x^2}{2\hbar}\right]$$
 (12)

which is the ground state of the simple harmonic oscillator.