Subtleties of Zero Modes

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Here, I introduce the two distinct zero mode components of the gauge field expansion and the subtlety of distinguishing their commutation relations.

I. FOURIER SERIES

As seen in "Generalizing the Soft Factor/Classical Connection," the time integral of F_{uz} gave the boundary term $N_z = A_z^+ - A_z^- = \partial_z N$. While this describes the zero mode behavior of the field strength, there is an additional part of the zero frequency behavior of A_z which is pure gauge, namely a constant-in-u shift $C_z = \frac{1}{2}[A_z^+ + A_z^-] = \partial_z C$. The value of C_z can change under a gauge transformation by an arbitrary function $\partial_z \lambda(z, \bar{z})$ of the angular variables, but what is important is that under such a u-independent gauge transformation, only C shifts while all other modes are unaffected. Since, N appears in the charge generating these large residual gauge transformations,[1] it is natural to look for two independent zero mode components which are conjugate to each other, while the bulk-bulk commutation relations that ordinarily appear in bracket formulations exclude the zero modes.

Consider performing an expansion over a finite interval $u \in [-\frac{T}{2}, \frac{T}{2}]$ of the gauge field along $\mathcal{I}^+$ for a particular point $(z, \bar{z})$ on the S^2:

$$\begin{aligned}\partial_u A_z &= a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{2\pi n}{T}u) + \sum_{n=1}^{\infty} b_n \sin(\frac{2\pi n}{T}u) \\ &= \sum_{n=-\infty}^{\infty} \alpha_n e^{\frac{i2\pi n}{T}u}\end{aligned} \tag{I.1}$$

where $\alpha_0 = a_0$, $\alpha_n = \frac{1}{2}(a_n - ib_n)$, $\alpha_{-n} = \frac{1}{2}(a_n + ib_n)$ for $n > 0$. The first line is shown to emphasize the presence of the constant term. Note that the above expansion assumes that the function is periodic on the interval $[-\frac{T}{2}, \frac{T}{2}]$. (It would be reasonable to consider the radiated electric field starting and ending at zero.) The coefficients are given by:

$$\alpha_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \partial_u A_z e^{-\frac{i2\pi n}{T}u} du. \tag{I.2}$$

Starting with (I.1) and integrating to find the expansion for A_z would give:

$$A_z = C_z + \alpha_0 u + \sum_{n \neq 0} \frac{T}{i2\pi n} \alpha_n e^{\frac{i2\pi n}{T}u}. \tag{I.3}$$

The subtlety of the zero mode, and the likely origin of discrepant factors of $\frac{1}{2}$ in matching residual gauge transformation commutators comes from the linear term. Using (I.3), we find $N_z = \alpha_0 T$. Meanwhile, If we performed a fourier series expansion of (I.3) on the interval $(-\frac{T}{2}, \frac{T}{2})$, we could absorb the linear term into the $\sin(\frac{2\pi n}{T}u)$ coefficients.

$$\begin{aligned}A_z' &= \sum_{n=-\infty}^{\infty} \alpha_n' e^{\frac{i2\pi n}{T}u} \\ \alpha_0' = C_z, \quad \alpha_n' &= \frac{iT}{2\pi n}[(-1)^n \alpha_0 - \alpha_n].\end{aligned} \tag{I.4}$$

This new expansion goes to C_z at both endpoints, while matching the function on the open interval.

II. FOURIER TRANSFORM

Taking the large T limit naturally leads to the Fourier transform when the mode coefficients are well behaved. Letting $\omega = \frac{2\pi n}{T} \to 0$ for $n \neq 0$,

$$\alpha_n = \frac{\omega}{2\pi n} \int_{-\infty}^{\infty} \partial_u A_z e^{-i\omega u} du. \tag{II.1}$$

Since $\omega = \frac{2\pi n}{T}$, when n increments by Δn, $d\omega = \frac{2\pi}{T}\Delta n = \frac{\omega}{n}\Delta n$ and $\sum\limits_{n=1}^{\infty} \frac{\omega}{n} ... \to \int_{0^+}^{\infty} d\omega...$, so that in this limit (I.1) becomes:

$$\partial_u A_z = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega u} \int_{-\infty}^{\infty} \partial_{u'} A_z e^{-i\omega u'} du'. \tag{II.2}$$

If we want to write:

$$A_z = \int_{-\infty}^{\infty} d\omega \alpha(\omega) e^{i\omega u}, \tag{II.3}$$

the mode expansion will include a $\delta(\omega)$ piece:

$$\alpha(\omega) = C_z \delta(\omega) + ... \tag{II.4}$$

not usually included in soft factor expansions since it sits at $\omega = 0$. For small ω the Weinberg pole behavior, proportional to N_z, dominates. This N_z appeared as a linear term in (I.3), however, a linearly growing term has an ill-defined Fourier transform. Since α_0 is suppressed by T^{-1} (the time integral rather than time average of F_{uz} gives the physically relevant quantity), the same limiting

[1] arXiv:1407.3789

behavior can be achieved with a sign function. Looking at:

$$\int_0^\infty \frac{2}{\pi\omega}\sin(\omega u)d\omega = \Theta(u) \tag{II.5}$$

shows how the ω^{-1} part of the mode expansion for A_z can pick out the N_z behavior. It is possible to modify the mode expansion with a sign function rather than a linear term by constructing:

$$\hat{A}_z = A_z - [\frac{N_z}{2}\Theta(u) + C_z] \tag{II.6}$$

which goes to zero on the boundaries. Note that the way in which N_z is split out of $\hat{A}_z$ can affect the $\mathcal{O}(\omega^{-1})$ behavior of what are defined as the new bulk modes. The downside of this choice is that the same limiting behavior occurs for translating Θ by finite Δu. However, the classical result for the massless case described in the previous paper showed that $\partial_u A_z$ was composed of $N_{zj}\delta(u-u_j)$ terms where the u_j corresponded to the timing of wavefronts of massless charged particles approaching $\mathcal{I}^+$ contributing to the full N_z.

When accelerations are assumed to take place over a long time frame, so that the radiated power is minimized while prescribed changes in velocities are still achieved, the Fourier series+linear expansion (I.3) has the appeal of describing a continuously radiating background, being the zero mode of the radiated field F_{uz}.

On the other hand, in the context of a single scattering process, where the interaction is centered at the spacetime origin so that the massless matter wavefront is centered at $u = 0$, an augmented mode expansion similar to (II.6) can be thought of as subtracting the classical radiation solution if all accelerations were forced to occur instantaneously, allowing the higher frequency modes to capture the fluctuations about this configuration in a way such that $\hat{A}_z$ decays to zero at both limits. From a QFT point of view, where the scattering occurs in a localized region, small compared to where the products would be detected, this second approach has a natural context.