Soft Shadows

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(Dated: March 3, 2017)

This note shows that generalizing the highest weight transform to include shadow terms can reproduce soft terms that decouple, in the corresponding $\lambda \to 0$ limit.

ISBN: 978-0-9863685-4-7

The final comments of [1] suggested a study of the second term in (I.4) of that note might have interesting implications. This note answers that curiosity in the affirmative. Previous treatments such as [2, 3] focused only on the Mellin term. Mellin amplitudes are still the building blocks for a treatment of both terms, and lead to transformed amplitudes with interesting features, worth studying in their own right [4]. Including both terms, however, offers the possibility of having non-contact low point functions, much like in the massive case (see A.5 of [5]). On its own, this impetus to then include the smeared terms just to get rid of the contact/singular structure might be hedged by the manner one trades the localization on the $\mathcal{CS}^2$ for a massless insertion (which seems like a rather nice property if we view the highest weight operators as sitting at the same point as where the massless particle punctures the celestial sphere at null infinity.

In this note, I demonstrate that such smearing is actually independently motivated by earlier work on soft theorems by A. Strominger et al [6]. Namely, a particular combination of the soft modes in electromagnetism decouples in amplitudes. As pointed out in A.4 of [6], the combination:

$$\mathbf{a}_- \equiv a_-(\omega\hat{x}) - \frac{1}{2\pi}\int d^2w \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}} a_+(\omega\hat{y})$$

has no poles coming from Weinberg's soft photon theorem. Note that here I have changed to coordinates relevant to comparison with [5]. From the 4D scattering perspective this reflects a relation between the two helicities in the soft limit of the radiation/memory effect [7]. In this note I show that the same combination arises from the $\lambda \to 0$ limit of a combination of Mellin+smeared solutions. Using that the $\lambda \to 0$ limit in Mellin space corresponds to the $\omega \to 0$ limit in momentum space [3] this implies that a certain Mellin+smeared combination decouples in the soft limit.

This note is organized as follows. First the decoupling of $\mathbf{a}_-$ is shown from evaluation of the soft theorem in section I. Next, in section II, the smeared solutions are shown to be CFT shadow operators of the conjugate weight and opposite helicity Mellin term for the spin 1 case. For the scalar case this was already seen in (I.4) of [1], however without the interpretation in terms of CFT shadows in that note. Section III shows how these shadow terms arise from a massless limit of the transform in [5] and how the mass dependence cancels in n-point functions in a way rather neat compared to the naive appearance of a mass in the transform itself. (Upshot: the sum over weight constraints for each term mean that the mass parameter always appears to the 0th power, i.e. the $m \to 0$ limit makes sense as long as it is taken after computing correlators rather than in the transform itself.) Section IV computes the two point functions between the basis operators that arose in previous sections.

I personally find it somewhat cute that the shadow smearing performs an involution among the spectrum of weights already appearing, i.e. if we have Mellin fields $\phi^+_{i\lambda}$ with $(h,\bar{h}) = (1+\frac{i\lambda}{2}, \frac{i\lambda}{2})$ and $\phi^-_{i\lambda}$ with $(h,\bar{h}) = (\frac{i\lambda}{2}, 1+\frac{i\lambda}{2})$ and we look for a second solution with weights $(h,\bar{h}) = (1+\frac{i\lambda}{2}, \frac{i\lambda}{2})$ aside from ϕ^+, the answer is precisely the shadow convolution of ϕ^-. This is just the statement that $(\frac{d}{2}-h, \frac{d}{2}-\bar{h})$, conjugates the weights and flips helicities, where here we have $d=2$. I also find it appealing that these shadow terms are directly motivated by the massive transform of [4].

It is important to note in this introduction that the shadow term is only $SL(2,\mathbb{C})$ covariant, i.e. covariant under only the global conformal group. When dealing with quasi-primaries this is all that is needed. However, this study of highest weight states was motivated by the manner in which they diagonalize the action of the subleading soft theorem [8] which motivated, as physical, a Virasoro action within the set of asymptotic symmetries of asymptotically flat spacetimes. If we were able to, one day, push this motivation to the full power of a CFT_2 with Virasoro symmetry, we would want to isolate the primary operators, not just quasi-primaries. The above statement about the shadow term transforming covariantly under only the global $SL(2,\mathbb{C})$ means that working out precisely which shadow terms appear as primaries will be an important point, eventually. i.e. the fact that the shadow term only preserves the quasi-primary transformations would imply that if a mixed Mellin+shadow term is the true conformal primary then the individual Mellin terms would not be. An alternative result could very well be that the Mellin operators are the primaries for certain fields, but that currents (like ex. the stress tensor [9]) are only quasi-primaries and have mixed Mellin+shadow terms. Thus, while one might be wary of just adding a smearing term, the manner in which this note shows that such a smearing is relevant to the modes that decouple from the soft photon theorem seems to also be relevant to the understanding of constraints between helicities in the soft limit, which in the soft gluon case led to non-commuting double soft limits and only the holomorphic current being independent in the Kac-Moody algebra that emerged from the soft/symmetry connection [10].

I. SOFT THEOREM AND DECOUPLING

Using the coordinate conventions of [5, 11], we have

$$q = \omega(1 + z\bar{z}, z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z}) \tag{I.1}$$

$$\epsilon^+ = \frac{1}{\sqrt{2}}(\bar{z}, 1, -i, -\bar{z}) \quad \epsilon^- = \frac{1}{\sqrt{2}}(z, 1, i, -z) \tag{I.2}$$

for the photon momentum and polarization vectors, as well as

$$p_k = \omega_k(1 + z_k\bar{z}_k, z_k + \bar{z}_k, -i(z_k - \bar{z}_k), 1 - z_k\bar{z}_k) \tag{I.3}$$

for the momenta of a massless charged particle. We can then evaluate the contribution to the Weinberg soft factor:

$$S^{(0)\pm} = \sum_k eQ_k \frac{p_k \cdot \epsilon^\pm}{p_k \cdot q}$$

coming from each charged particle in an amplitude onto which we've added a soft photon:

$$S_k^{(0)+} = \frac{eQ_k}{\sqrt{2}\omega}\frac{1}{z - z_k} \quad S_k^{(0)-} = \frac{eQ_k}{\sqrt{2}\omega}\frac{1}{\bar{z} - \bar{z}_k}. \tag{I.4}$$

Then, using $\bar{\partial}\frac{1}{z} = 2\pi\delta^2(z, \bar{z})$, we see that the combination

$$\mathbf{a}_- \equiv a_-(\omega\hat{x}) - \frac{1}{2\pi}\int d^2w \frac{1}{\bar{z} - \bar{w}}\partial_{\bar{w}} a_+(\omega\hat{y}) \tag{I.5}$$

with $\hat{x}$ parameterized by z, $\hat{y}$ by w, decouples in the soft limit, as in [6]. For highest weight scattering in (I.6) of [1] we defined Mellin transformed annhilation operators:

$$\mathfrak{a}_\lambda^\pm(\hat{q}) \equiv \int_0^\infty d\omega\ \omega^{i\lambda} a^\pm(\omega\hat{q}). \tag{I.6}$$

Since this transform only acts on the energy dependence, the un-smeared and smeared terms in $\mathbf{a}_-$ persist in the mellin transformed modes. From [3] we see that the soft momentum limit corresponds to the $\lambda \to 0$ limit, so we expect that a soft decoupling would match the mellin transform of a in the $\lambda \to 0$ limit. i.e. the same pure Mellin$^-$+smeared Mellin$^+$ combination will decouple when $\lambda \to 0$. We will see in the next section from $SL(2,\mathbb{C})$ consistency of a highest weight solution, that the appropriate Mellin combination is not exactly just Mellin transform of $\mathbf{a}$ but indeed can reduce to this form as $\lambda \to 0$ (in particular, we will find that the weights are conjugated and that the integral transform also has a λ dependence).

II. SHADOW TERM FROM $SL(2,\mathbb{C})$

We can think of the Mellin transform as a map

$$M : \langle 0|a_\pm(\omega\hat{q})...|0\rangle \to \langle\phi_{i\lambda}^\pm...\rangle \tag{II.1}$$

to correlators of $\phi_{i\lambda}^+$ with $(h,\bar{h}) = (1 + \frac{i\lambda}{2}, \frac{i\lambda}{2})$ and $\phi_{i\lambda}^-$ with $(h,\bar{h}) = (\frac{i\lambda}{2}, 1 + \frac{i\lambda}{2})$ that correspond to $\pm$-helicity (outgoing) photon states. In some since this is almost just a relabeling of $\mathfrak{a}_\lambda^\pm$ but as is seen in [4] there is a natural kinematic $in-out$ crossing which arises in these Mellin amplitudes so using $\phi_{i\lambda}^\pm$ distances the attachment to only a creation or annhiliation operator (the helicity here would be the same as in the all-out formalism). If we demand that under $SL(2,\mathbb{C})$:

$$\phi'(w', \bar{w}') = \left(\frac{dw'}{dw}\right)^{-h}\left(\frac{d\bar{w}'}{d\bar{w}}\right)^{-\bar{h}}\phi(w,\bar{w}) \tag{II.2}$$

and that this holds for the above $\phi_{i\lambda}^\pm$ for the prescribed weights, then we find that these two solutions are not the most arbitrary quasi-primaries with the specified weights $(h,\bar{h}) = (1 + \frac{i\lambda}{2}, \frac{i\lambda}{2})$ or $(h,\bar{h}) = (\frac{i\lambda}{2}, 1 + \frac{i\lambda}{2})$ that can be built out of them.

Namely, the shadow convolution:

$$\tilde{\phi}(z,\bar{z}) = K_{h,\bar{h}}\frac{1}{2\pi}\int d^2w \frac{\phi(w,\bar{w})}{(z - w)^{2-2h}(\bar{z} - \bar{w})^{2-2\bar{h}}} \tag{II.3}$$

has weight $(\tilde{h}, \tilde{\bar{h}}) = (1-h, 1-\bar{h})$. (Here I have used (2)-(3) from [12], where $K_{h,\bar{h}} = \frac{\Gamma[2-2\bar{h}]}{\Gamma[2h-1]}$ gives $\tilde{\tilde{\phi}} = (-1)^{2(h-\bar{h})}\phi$, there is a factor of 2 relative to Osborn's definition coming from $d^2w = d\mathrm{Re}[z]d\mathrm{Im}[z]$ for his conventions as opposed to twice this when we write it here to be consistent with our normalzation for $\bar{\partial}\frac{1}{z}$ as was used in deriving (I.5).) We thus see that $\tilde{\phi}_{-i\lambda}^+$ has the same weight as $\phi_{i\lambda}^-$ and $\tilde{\phi}_{-i\lambda}^-$ has the same weight as $\phi_{i\lambda}^+$. However these are not the same operators, as positive and negative helicity photons are independent (at least at non-zero frequency).

We can thus define generalized highest weight solutions:

$$\mathcal{O}_{i\lambda}^+(z,\bar{z}) = \phi_{i\lambda}^+(z,\bar{z}) + C_{+,\lambda}\int d^2w \frac{1}{(w-z)^{2+i\lambda}(\bar{w}-\bar{z})^{i\lambda}}\phi_{-i\lambda}^-(w,\bar{w})$$

$$\mathcal{O}_{i\lambda}^-(z,\bar{z}) = \phi_{i\lambda}^-(z,\bar{z}) + C_{-,\lambda}\int d^2w \frac{1}{(w-z)^{i\lambda}(\bar{w}-\bar{z})^{2+i\lambda}}\phi_{-i\lambda}^+(w,\bar{w}) \tag{II.4}$$

for some constants $C_{\pm,\lambda}$. $\mathcal{O}_{i\lambda}^+$ and $\mathcal{O}_{i\lambda}^-$ also have weights $(h,\bar{h}) = (1+\frac{i\lambda}{2}, \frac{i\lambda}{2})$ and $(h,\bar{h}) = (\frac{i\lambda}{2}, 1+\frac{i\lambda}{2})$, respectively.

II.1. Soft Decoupling with Shadows

Let us now look at the $\lambda \to 0$ limit of $\mathcal{O}_{i\lambda}^-$:

$$\mathcal{O}_0^-(z,\bar{z}) = \phi_0^-(z,\bar{z}) + C_{-,0}\int d^2w \frac{1}{(\bar{w} - \bar{z})^2}\phi_0^+(w,\bar{w}) \tag{II.5}$$

If we integrate by parts with respect to $\partial_{\bar{w}}$ in (I.5) we get

$$\mathbf{a}_- = a_-(\omega\hat{x}) + \frac{1}{2\pi}\int d^2w \frac{1}{(\bar{w} - \bar{z})^2}a_+(\omega\hat{y}). \tag{II.6}$$

With the $\lambda \to 0/\omega \to 0$ correspondence of [3] we thus get that $\mathcal{O}_0^-$ decouples from correlation functions if $C_{-,\lambda}$ is such that $C_{-,0} = \frac{1}{2\pi}$. This connection between [6] and potential shadow terms is what motivated this note.

Plugging in the relevant values for $(h, \bar{h})$ for $K_{h,\bar{h}}$ when $\lambda \to 0$, we see that $C_{-,0} = \frac{1}{2\pi}$ corresponds to the combination

$$\mathcal{O}_0^-(z,\bar{z}) = \phi_0^-(z,\bar{z}) + \tilde{\phi}_0^+(z,\bar{z}). \tag{II.7}$$

This is the combination that decouples in correlators (note that here I am not excluding an overall λ dependent normalization that might be required before taking the $\lambda \to 0$ limit to ex. pick out residues. The statement is about the relative coefficient as $\lambda \to 0$ and that from (I.5) the residue of the Weinberg pole should be zero). Since for integer spin $\tilde{\tilde{\phi}} = \phi$ we get that

$$\tilde{\mathcal{O}}_0^-(z,\bar{z}) = \tilde{\phi}_0^-(z,\bar{z}) + \phi_0^+(z,\bar{z}) \equiv \mathcal{O}_0^+(z,\bar{z}) \tag{II.8}$$

would naturally correspond to the soft limit of $\mathbf{a}_+$ – the positive helicity version of (I.5) – which would also decouple from the same soft theorem manipulations that led to (I.5). Note we have thus found that

$$\boxed{\mathcal{O}_{i\lambda}^{\pm}(z,\bar{z}) \equiv f(i\lambda)\phi_{i\lambda}^{\pm}(z,\bar{z}) + f(-i\lambda)\tilde{\phi}_{-i\lambda}^{\mp}(z,\bar{z})} \tag{II.9}$$

are shadow conjugates $\tilde{\mathcal{O}}_{i\lambda}^{\pm} = \mathcal{O}_{-i\lambda}^{\mp}$ and decouple in their $\lambda \to 0$ limit. It seems to me like a rather nice feature that this decoupling a la Weinberg's soft theorem is also closed under taking shadows. Any combination that reduces to one of these as $\lambda \to 0$ would also decouple, however replacing the relative coefficient with a more general $g(i\lambda)$ in place of $\frac{f(-i\lambda)}{f(i\lambda)}$ such that $g(0) = 1$ would ruin the closure-under shadows that these two operator families obey for all λ in (III.24).

Note although the decoupled fields play an interesting role and motivate the relevance of shadow solutions, nothing prevents us from restricting ourselves to linear combinations that do decouple. For a complete set of solutions would likely expect a different linear combination of operators. In the following section, we will see what combinations naturally arise from a massless limit of the transform in [5].

III. SHADOW TERM FROM $m \to 0$

In position space, there are two highest weight scalar solutions to the massless Klein-Gordon equation:

$$\phi_\Delta^{\pm} = a\frac{1}{(-q\cdot X \pm i\epsilon)^\Delta} + b\frac{(X^2)^{\Delta-1}}{(-q\cdot X \pm i\epsilon)^\Delta} \tag{III.1}$$

the first term corresponds to the Mellin transform of [2, 3]. When $\mathrm{Re}[\Delta] = 1$ the massless limit of the position space solution corresponding to the momentum space bulk-to-boundary propagator does not select only this term, but rather a combination of both. In this section, we will explore how both the Mellin and its shadow arise from the massless limit of the vector bulk-to-boundary propagator. Taking the massless limit is one way to fix the arbitrary coefficients in (II.4) for solutions with the appropriate $SL(2,\mathbb{C})$ covariance. That is what we shall do here, so that we can, afterwards, compare these to the coefficients for the modes that decouple in the soft limit (III.24).

If we look for massive spin 1 solutions that are Lorentz vectors with respect to X^μ and spin 1 conformal quasi-primaries with respect to $(w, \bar{w})$, then we find that the corresponding bulk to boundary propagators are:

$$\begin{aligned} G^\Delta_{\mu w}(y,z,\bar{z};w,\bar{w}) &= \partial_w k_\mu \left(\tfrac{y}{y^2+|z-w|^2}\right)^\Delta + k_\mu \tfrac{\bar{z}-\bar{w}}{y}\left(\tfrac{y}{y^2+|z-w|^2}\right)^{\Delta+1} \\ G^\Delta_{\mu \bar{w}}(y,z,\bar{z};w,\bar{w}) &= \partial_{\bar{w}} k_\mu \left(\tfrac{y}{y^2+|z-w|^2}\right)^\Delta + k_\mu \tfrac{z-w}{y}\left(\tfrac{y}{y^2+|z-w|^2}\right)^{\Delta+1} \end{aligned} \tag{III.2}$$

thanks to S.H. Shao [4]. Here $(y, z, \bar{z})$ parameterize a unit hyperboloid:

$$\hat{p}^\mu(y,z,\bar{z}) = \left(\frac{1+y^2+|z|^2}{2y}, \frac{\mathrm{Re}(z)}{y}, \frac{\mathrm{Im}(z)}{y}, \frac{1-y^2-|z|^2}{2y}\right), \tag{III.3}$$

corresponding to an on-shell massive four momentum a la [5], and k is a reference null vector

$$k = (1 + w\bar{w}, w + \bar{w}, -i(w-\bar{w}), 1 - w\bar{w}) \tag{III.4}$$

so that

$$\partial_w k_\mu = (\bar{w}, 1, -i, -\bar{w}), \quad \partial_{\bar{w}} k_\mu = (w, 1, i, -w) \tag{III.5}$$

match what would be polarization tensors for a massless momentum in the direction of $(w, \bar{w})$ on the $\mathcal{CS}^2$. The massive polarizations would here depend on $(y, z, \bar{z})$:

$$\begin{aligned} \epsilon^L &= \left(\tfrac{1-y^2+|z|^2}{2y}, \tfrac{\mathrm{Re}(z)}{y}, \tfrac{\mathrm{Im}(z)}{y}, \tfrac{1+y^2-|z|^2}{2y}\right) \\ \epsilon^+ &= \sqrt{2}y\partial_z\hat{p} = \tfrac{1}{\sqrt{2}}(\bar{z}, 1, -i, -\bar{z}) \\ \epsilon^- &= \sqrt{2}y\partial_{\bar{z}}\hat{p} = \tfrac{1}{\sqrt{2}}(z, 1, i, -z) \end{aligned} \tag{III.6}$$

where ϵ^L is solved for by demanding that it have unit norm and be orthogonal to $(\hat{p}, \epsilon^+, \epsilon^-)$. Note that ϵ^L differs from $\hat{p}$ by a change in sign of the y^2 term, so one can nicely see that it approaches $\hat{p}$ in the boundary limit where $y \to 0$, matching how a null hypersurface is orthogonal to its null generator. The nonzero inner products among these polarization tensors are:

$$\epsilon^L \cdot \epsilon^L = 1 \quad \epsilon^+ \cdot \epsilon^- = 1 \tag{III.7}$$

We also care about their dot products with $(k, \partial_w k, \partial_{\bar{w}} k)$:

$$\begin{array}{lll} \epsilon^L \cdot k = \frac{y^2-|w-z|^2}{y} & \epsilon^- \cdot k = \sqrt{2}(w-z) & \epsilon^+ \cdot k = \sqrt{2}(\bar{w}-\bar{z}) \\ \epsilon^L \cdot \partial_w k = \frac{\bar{z}-\bar{w}}{y} & \epsilon^- \cdot \partial_w k = \sqrt{2} & \epsilon^+ \cdot \partial_w k = 0 \\ \epsilon^L \cdot \partial_{\bar{w}} k = \frac{z-w}{y} & \epsilon^- \cdot \partial_{\bar{w}} k = 0 & \epsilon^+ \cdot \partial_{\bar{w}} k = \sqrt{2} \end{array} \tag{III.8}$$

As a result:

$$\begin{aligned} \epsilon^L \cdot G^\Delta_w(y,z,\bar{z};w,\bar{w}) &= 2(\bar{z}-\bar{w})\left(\tfrac{y}{y^2+|z-w|^2}\right)^{\Delta+1} \\ \epsilon^- \cdot G^\Delta_w(y,z,\bar{z};w,\bar{w}) &= \sqrt{2}y\left(\tfrac{y}{y^2+|z-w|^2}\right)^{\Delta+1} \\ \epsilon^+ \cdot G^\Delta_w(y,z,\bar{z};w,\bar{w}) &= -\tfrac{\sqrt{2}(\bar{z}-\bar{w})^2}{y}\left(\tfrac{y}{y^2+|z-w|^2}\right)^{\Delta+1} \end{aligned} \tag{III.9}$$

and

$$\begin{aligned}\epsilon^L\cdot G^\Delta_{\bar w}(y,z,\bar z;w,\bar w) &= 2(z-w)\left(\tfrac{y}{y^2+|z-w|^2}\right)^{\Delta+1}\\ \epsilon^-\cdot G^\Delta_{\bar w}(y,z,\bar z;w,\bar w) &= -\tfrac{\sqrt2(z-w)^2}{y}\left(\tfrac{y}{y^2+|z-w|^2}\right)^{\Delta+1}\\ \epsilon^+\cdot G^\Delta_{\bar w}(y,z,\bar z;w,\bar w) &= \sqrt2 y\left(\tfrac{y}{y^2+|z-w|^2}\right)^{\Delta+1}\end{aligned} \tag{III.10}$$

Note that the replacement $y=\frac{m}{2\omega}$, holding ω fixed, implies that we want to look at the behavior of these solutions at small y. Using the result from the scalar case, we find.

$$\begin{aligned}\epsilon^L\cdot G^\Delta_{w}(y,z,\bar z;w,\bar w) &\sim 2(\bar z-\bar w)\tfrac{y^{\Delta+1}}{|z-w|^{2(\Delta+1)}}+...\\ \epsilon^-\cdot G^\Delta_{w}(y,z,\bar z;w,\bar w) &\sim \sqrt2[C_{\Delta+1}y^{2-\Delta}\delta^2(z-w)+\tfrac{y^{\Delta+2}}{|z-w|^{2(\Delta+1)}}...]\\ \epsilon^+\cdot G^\Delta_{w}(y,z,\bar z;w,\bar w) &\sim -\sqrt2(\bar z-\bar w)^2\tfrac{y^{\Delta}}{|z-w|^{2(\Delta+1)}}+...\end{aligned} \tag{III.11}$$

and

$$\begin{aligned}\epsilon^L\cdot G^\Delta_{\bar w}(y,z,\bar z;w,\bar w) &\sim 2(z-w)\tfrac{y^{\Delta+1}}{|z-w|^{2(\Delta+1)}}+...\\ \epsilon^-\cdot G^\Delta_{\bar w}(y,z,\bar z;w,\bar w) &\sim -\sqrt2(z-w)^2\tfrac{y^{\Delta}}{|z-w|^{2(\Delta+1)}}+...\\ \epsilon^+\cdot G^\Delta_{\bar w}(y,z,\bar z;w,\bar w) &\sim \sqrt2[C_{\Delta+1}y^{2-\Delta}\delta^2(z-w)+\tfrac{y^{\Delta+2}}{|z-w|^{2(\Delta+1)}}...]\end{aligned} \tag{III.12}$$

where the contact terms are killed if multiplied by powers of $(z-w)$ or $(\bar z-\bar w)$ which vanish at the locus of their support. With $\Delta=1+i\lambda$, keeping only the leading y terms we see

$$\begin{aligned}\epsilon^L\cdot G^\Delta_{w}(y,z,\bar z;w,\bar w) &\sim 0\\ \epsilon^-\cdot G^\Delta_{w}(y,z,\bar z;w,\bar w) &\sim \sqrt2 C_{\Delta+1}y^{1-i\lambda}\delta^2(z-w)\\ \epsilon^+\cdot G^\Delta_{w}(y,z,\bar z;w,\bar w) &\sim -\sqrt2\tfrac{y^{1+i\lambda}}{(z-w)^{2+i\lambda}(\bar z-\bar w)^{i\lambda}}\end{aligned} \tag{III.13}$$

and

$$\begin{aligned}\epsilon^L\cdot G^\Delta_{\bar w}(y,z,\bar z;w,\bar w) &\sim 0\\ \epsilon^-\cdot G^\Delta_{\bar w}(y,z,\bar z;w,\bar w) &\sim -\sqrt2\tfrac{y^{1+i\lambda}}{(z-w)^{i\lambda}(\bar z-\bar w)^{2+i\lambda}}\\ \epsilon^+\cdot G^\Delta_{\bar w}(y,z,\bar z;w,\bar w) &\sim \sqrt2 C_{\Delta+1}y^{1-i\lambda}\delta^2(z-w)\end{aligned} \tag{III.14}$$

or, using polarization completeness relations (and transversality of the bulk to boundary propagator):

$$\begin{aligned}G^\Delta_{\mu w}(y,z,\bar z;w,\bar w) &\sim \sqrt2[\epsilon^+_\mu C_{\Delta+1}y^{1-i\lambda}\delta^2(z-w)-\epsilon^-_\mu\tfrac{y^{1+i\lambda}}{(z-w)^{2+i\lambda}(\bar z-\bar w)^{i\lambda}}]\\ G^\Delta_{\mu\bar w}(y,z,\bar z;w,\bar w) &\sim \sqrt2[\epsilon^-_\mu C_{\Delta+1}y^{1-i\lambda}\delta^2(z-w)-\epsilon^+_\mu\tfrac{y^{1+i\lambda}}{(z-w)^{i\lambda}(\bar z-\bar w)^{2+i\lambda}}].\end{aligned} \tag{III.15}$$

The origins of shadow terms appear in the non-contact terms that remain, in a manner similar to the scalar case considered in [1] but with the interesting feature that the shadow of the opposite helicity compared to the contact term appears (as it ought to be from $SL(2,\mathbb{C})$ invariance, but it is nice to see how it works out in this manner, i.e. there is either a contact term or smeared term surviving the massless limit for different polarization projections).

The longitudinal modes appear to decouple. While one might want to further consider the effect of gauge redundancy in the massless theory, the way in which the mode that limits to the unphysical polarization decouples seems promising to the hope that there is a sense in which our search for $\phi^\pm_{i\lambda}$ is selecting out the two physical helicity states and that these physical modes are what survives in the massless limit taken above. A more detailed exposition of this will be in a followup.

Considering the positive frequency part of the massive gauge field mode expansion:

$$\begin{aligned}&A^+_\mu(X)=\tfrac{m^3}{(2\pi)^3}\int_0^\infty\tfrac{dy}{y^3}\int dzd\bar z e^{im\hat p\cdot X}\\ &\times[\epsilon^{*L}_\mu a_L(y,z,\bar z;m)+\epsilon^{*-}_\mu a_-(y,z,\bar z;m)+\epsilon^{*+}_\mu a_+(y,z,\bar z;m)]\end{aligned} \tag{III.16}$$

Performing an inverse fourier transform in position space at $t=0$ would give

$$\begin{aligned}\hat A^+_\mu(y,z,\bar z) &= \int_{t=0}d^3Xe^{im\hat p\cdot X}A^+_\mu(X)\\ &= [\epsilon^L_\mu a_L(y,z,\bar z;m)+\epsilon^+_\mu a_-(y,z,\bar z;m)+\epsilon^-_\mu a_+(y,z,\bar z;m)]\end{aligned} \tag{III.17}$$

where we have used $\epsilon^{*L}_\mu=\epsilon^L_\mu$, and $\epsilon^{*\pm}_\mu=\epsilon^\mp_\mu$. We can then define annihilation operators for two highest weight states as

$$\begin{aligned}\mathfrak{a}^\Delta_w(w,\bar w) &\equiv \tfrac{m}{2\sqrt2}\int_0^\infty\tfrac{dy}{y^3}\int dzd\bar z\; G^\Delta_w(y,z,\bar z;w,\bar w)\cdot\hat A^+(y,z,\bar z)\\ \mathfrak{a}^\Delta_{\bar w}(w,\bar w) &\equiv \tfrac{m}{2\sqrt2}\int_0^\infty\tfrac{dy}{y^3}\int dzd\bar z\; G^\Delta_{\bar w}(y,z,\bar z;w,\bar w)\cdot\hat A^+(y,z,\bar z)\end{aligned} \tag{III.18}$$

That transform like spin ±1, respectively, under $SL(2,\mathbb{C})$. Note that because we care about the massless limit, we are not concerned about the apparent reduction in degrees of freedom (perhaps if we were interested in a massive vector field itself, we could capture the longitudinal mode with a bulk to boundary propagator that is a spacetime vector but an $SL(2,\mathbb{C})$ scalar, which would match the intuition of this polarization pointing normal to the $\mathcal{CS}^2$, and would also have been projected out in the above computations).

Taking the massless limit via $y=\frac{m}{2\omega}$ and assuming that we can take $a_\pm(y,z,\bar z;m)$ to $a_\pm(z,\bar z;\omega)$, the annihiliation operators for onshell massless photons of helicity ±1 and energy ω (the a_L was seen to decouple and from (III.3) $m\hat p$ would limit to a massless four-momentum with frequency parameter ω up to quadratic in y corrections), we get

$$\begin{aligned}\mathfrak{a}^\Delta_w(w,\bar w)\to &\int_0^\infty d\omega\int dzd\bar z[\left(\tfrac{m}{2}\right)^{-i\lambda}C_{\Delta+1}\delta^2(z-w)\omega^{i\lambda}a_+(z,\bar z;\omega)\\ &-\left(\tfrac{m}{2}\right)^{i\lambda}\tfrac{1}{(z-w)^{2+i\lambda}(\bar z-\bar w)^{i\lambda}}\omega^{-i\lambda}a_-(z,\bar z;\omega)]\\ &=\left(\tfrac{m}{2}\right)^{-i\lambda}C_{\Delta+1}\mathfrak{a}^+_{i\lambda}-\left(\tfrac{m}{2}\right)^{i\lambda}\int dzd\bar z\tfrac{1}{(z-w)^{2+i\lambda}(\bar z-\bar w)^{i\lambda}}\mathfrak{a}^-_{-i\lambda}\end{aligned} \tag{III.19}$$

Similarly

$$\begin{aligned}\mathfrak{a}^\Delta_{\bar w}(w,\bar w)\to &\int_0^\infty d\omega\int dzd\bar z[\left(\tfrac{m}{2}\right)^{-i\lambda}C_{\Delta+1}\delta^2(z-w)\omega^{i\lambda}a_-(z,\bar z;\omega)\\ &-\left(\tfrac{m}{2}\right)^{i\lambda}\tfrac{1}{(z-w)^{2+i\lambda}(\bar z-\bar w)^{i\lambda}}\omega^{-i\lambda}a_+(z,\bar z;\omega)]\\ &=\left(\tfrac{m}{2}\right)^{-i\lambda}C_{\Delta+1}\mathfrak{a}^-_{i\lambda}-\left(\tfrac{m}{2}\right)^{i\lambda}\int dzd\bar z\tfrac{1}{(z-w)^{i\lambda}(\bar z-\bar w)^{2+i\lambda}}\mathfrak{a}^+_{-i\lambda}\end{aligned} \tag{III.20}$$

So we see that both contact and smeared terms arise. However, one might be rather off-put by the appearance of powers of $\left(\frac{m}{2}\right)^{\pm i\lambda}$ when we claimed we were taking the $m\to0$ limit and used this to justify looking at only the boundary behavior of the bulk-to-boundary momentum space propagators. What gives? Actually there is a rather intriguing result. Although the two terms have opposite phases in their mass dependence, so that we can't extract one by taking the limit of $m\to0$ after multiplying the transform by a power of m, the mass dependence cancels in correlation functions. Namely, in any correlation function the sum over λ_i vanishes. Since the power of the mass is always opposite the operator it appears next to, i.e. $\left(\frac{m}{2}\right)^{-i\lambda}\mathfrak{a}^{\pm}_{i\lambda}$ or $\left(\frac{m}{2}\right)^{i\lambda}\mathfrak{a}^{\pm}_{-i\lambda}$, this guarantees that the net power of mass in any correlator vanishes as long as all fields within the correlator use the same mass parameter as a regulator (this last point might be an interesting one to pursue later, i.e. if there are other highest weight fields that are massive, what happens when only the total sum over weights is zero and not the individual sums for the massless sector).

Incorporating the mass cancellation in correlators, the above motivates us to define the following operators as the massless limit of the massive vector field:

$$\begin{aligned}\Phi^+_{i\lambda}(z,\bar z) &= c_\lambda[\phi^+_{i\lambda}(z,\bar z) - C^{-1}_{\Delta+1}\int d^2w \tfrac{1}{(w-z)^{2+i\lambda}(\bar w-\bar z)^{i\lambda}}\phi^-_{-i\lambda}(w,\bar w)]\\ \Phi^-_{i\lambda}(z,\bar z) &= c_\lambda[\phi^-_{i\lambda}(z,\bar z) - C^{-1}_{\Delta+1}\int d^2w \tfrac{1}{(w-z)^{i\lambda}(\bar w-\bar z)^{2+i\lambda}}\phi^+_{-i\lambda}(w,\bar w)]\end{aligned} \tag{III.21}$$

which have exactly the same form as (II.4) but with the relative constant fixed by the massless limit an overall normalization constant c_λ added for generality. From (11) of [13], taking into account our d^2z measure we find $C^{-1}_{\Delta+1} = \frac{\Gamma[\Delta+1]}{2\pi\Gamma[\Delta]} = \frac{\Gamma[2+i\lambda]}{2\pi\Gamma[1+i\lambda]}$. This implies:

$$\begin{aligned}\Phi^+_{i\lambda}(z,\bar z) &= c_\lambda[\phi^+_{i\lambda}(z,\bar z) - \tfrac{\Gamma[1-i\lambda]}{\Gamma[1+i\lambda]}\tilde\phi^-_{-i\lambda}(w,\bar w)]\\ \Phi^-_{i\lambda}(z,\bar z) &= c_\lambda[\phi^-_{i\lambda}(z,\bar z) - \tfrac{\Gamma[1-i\lambda]}{\Gamma[1+i\lambda]}\tilde\phi^+_{-i\lambda}(w,\bar w)]\end{aligned} \tag{III.22}$$

This suggests a change in normalization for our modes (which just multiplies both helicity operators by the same λ-dependent constant in the new definition of the Mellin transform). Let us define:

$$\varphi^+_{i\lambda}(z,\bar z) \equiv \Gamma[1+i\lambda]\phi^+_{i\lambda}(z,\bar z) \tag{III.23}$$

and let $c_\lambda = i\Gamma[1+i\lambda]$ then the combinations:

$$\boxed{\Phi^{\pm}_{i\lambda}(z,\bar z) \equiv i[\varphi^{\pm}_{i\lambda}(z,\bar z) - \tilde\varphi^{\mp}_{-i\lambda}(z,\bar z)]} \tag{III.24}$$

are the massless limits of the two helicity states for the vector analog of the [5] transform. This normalization also suggests a natural choice of $f(i\lambda)$ in (III.24) so that

$$\boxed{\mathcal{O}^{\pm}_{i\lambda}(z,\bar z) \equiv \varphi^{\pm}_{i\lambda}(z,\bar z) + \tilde\varphi^{\mp}_{-i\lambda}(z,\bar z)} \tag{III.25}$$

are the combinations that decouple in the soft limit. Note that as $\lambda\to 0$, $\Gamma[1-i\lambda]\to 1$ and the modified normalization approaches unity in this limit. It is curious how both the massless limit and the decoupled modes are separately closed under shadow operation:

$$\boxed{\begin{aligned}\tilde\Phi^{\pm}_{i\lambda}(z,\bar z) &= -\Phi^{\mp}_{-i\lambda}(z,\bar z)\\ \tilde{\mathcal{O}}^{\pm}_{i\lambda}(z,\bar z) &= \mathcal{O}^{\mp}_{-i\lambda}(z,\bar z)\end{aligned}} \tag{III.26}$$

Here, the factor of i in c_λ was motivated a posteriori by preferring positive two point functions in the following section. It would be interesting to understand better the connection to hermiticity. In a full treatment of this aspect one would need to delve more deeply into the *in-out* crossed amplitudes. A rather intriguing story has emerged in this respect in [4], in which the Mellin correlators are naturally a sum over all crossings. In that sense while we have $a^\dagger_-$ and a_+ both corresponding to ϕ^+ insertions, one must be careful in defining exactly what combination we want to appear. If we take quite literally the sum over all crossings, this would be like taking the sum of creation and annihilation operators. i.e. $(\mathfrak{a}^+_{i\lambda} + (\mathfrak{a}^-_{-i\lambda})^\dagger)^\dagger = \mathfrak{a}^-_{-i\lambda} + (\mathfrak{a}^+_{i\lambda})^\dagger$ so that if these were the combinations appearing in $\varphi^{\pm}_{i\lambda}$ then $(\varphi^{\pm}_{i\lambda})^\dagger = \varphi^{\mp}_{-i\lambda}$ the i in c_λ would be required for hermitian conjugation to correspond to shadow operation for both $\Phi^{\pm}_{i\lambda}$ and $\mathcal{O}^{\pm}_{i\lambda}$. Is this natural? The fact that hermiticity would be related for a positive two point coefficient would not come as a surprise.

I would not call this last observation a derivation, but it is fun to appreciate how consistency seems to suggest answers to questions that have been present for some time. For instance, I find it cute that many of the subtleties of the zero mode analysis of [6, 8] surrounded precisely how to combine positive and negative frequencies in the zero energy limit (i.e. the symmetry generating operators there are sums of creation and annihilation operators as the energy is sent to zero). In light of the Mellin analysis [5] perhaps this combination is what is natural want for all λ not just the soft limit.

Note that as a linear operator the shadow convolution is diagonalized by the combinations:

$$\begin{array}{lcl}\mathcal{O}^+_{i\lambda}(z,\bar z)+\mathcal{O}^-_{-i\lambda}(z,\bar z) &\mapsto& \mathcal{O}^+_{i\lambda}(z,\bar z)+\mathcal{O}^-_{-i\lambda}(z,\bar z)\\ \mathcal{O}^+_{i\lambda}(z,\bar z)-\mathcal{O}^-_{-i\lambda}(z,\bar z) &\mapsto& -\left[\mathcal{O}^+_{i\lambda}(z,\bar z)-\mathcal{O}^-_{-i\lambda}(z,\bar z)\right]\\ \Phi^+_{i\lambda}(z,\bar z)-\Phi^-_{-i\lambda}(z,\bar z) &\mapsto& \Phi^+_{i\lambda}(z,\bar z)-\Phi^-_{-i\lambda}(z,\bar z)\\ \Phi^+_{i\lambda}(z,\bar z)+\Phi^-_{-i\lambda}(z,\bar z) &\mapsto& -\left[\Phi^+_{i\lambda}(z,\bar z)+\Phi^-_{-i\lambda}(z,\bar z)\right]\end{array} \tag{III.27}$$

with eigenvalues 1 and -1, which essentially amounts to having mellin and shadow combinations appear together. These mix $SL(2,\mathbb{C})$ weights and so are not the combinations we are after. From the perspective of this note, the separate closure of the soft-decoupled versus $m\to 0$ limit terms is what is most intriguing. The interpretation may be related to the manner in which the decoupled mode could be connected to gauge transformations, whereas the massless limit will imply a certain gauge choice. Though this gauge choice should not affect the physical degrees of freedom, it would be curious to explore further the connection to the soft/symmetry story. Since the $\Phi^{\pm}_{i\lambda}$ combinations do not decouple, it is their soft limit that probes the Weinberg pole and would generate the current algebra on the $\mathcal{CS}^2$. There is still physical radiation (ex the memory effect [7]) corresponding to this zero energy limit, hence we expect we could pick this up even with gauge fixing. Meanwhile, the constraint between polarization at zero energy is what causes the $\mathcal{O}^{\pm}_{i\lambda}$ modes to decouple.

What about degree of freedom counting? We should be able to capture everything with just the Mellin modes $\varphi^{\pm}_{i\lambda}$ so while this note suggests a natural change of basis, we haven't quite made this choice yet with regards to what operators we want to use as our two replacements for $\varphi^{\pm}_{i\lambda}$. Since shadow correlations can be determined by convolution, their correlators are not independent. It would then seem like we would want to choose only one helcity$\Phi\,_{i\lambda}$ and one for $\mathcal{O}_{i\lambda}$. Ex. If we pick as independent$\Phi\,^{+}_{i\lambda}$ and $\mathcal{O}^{+}_{i\lambda}$ we can isolate the mellin states via:

$$\varphi^{+}_{i\lambda} = \frac{1}{2}\left[-i\Phi^{+}_{i\lambda} + \mathcal{O}^{+}_{i\lambda}\right] \tag{III.28}$$

and taking the shadow of

$$\tilde{\varphi}^{-}_{i\lambda} = \frac{1}{2}\left[i\Phi^{+}_{i\lambda} + \mathcal{O}^{+}_{i\lambda}\right] \tag{III.29}$$

since $\tilde{\tilde{\varphi}}^{-}_{i\lambda} = \varphi^{-}_{i\lambda}$. Choosing $\{\Phi^{-}_{i\lambda}, \mathcal{O}^{-}_{i\lambda}\}$ would just flip which correlators are easier to compute from our basis (+ vs − needing a shadow). A choice of opposite helicities like $\{\Phi^{+}_{i\lambda}, \mathcal{O}^{-}_{i\lambda}\}$ would be mathematically equivalent since we can still write $\varphi^{\pm}_{i\lambda}$ as a linear combination of one of these and the shadow of the other, but somehow seems less natural for correlation functions since when the basis is the same helicity, you either shadow transform or leave alone a given coordinate when you write everything in terms of $\{\Phi^{+}_{i\lambda}, \mathcal{O}^{+}_{i\lambda}\}$ correlators. With the mixed basis you would have a sum of two operators at the same point one of which you shadow transform, the other not.

In this manner, it is curious how closure under shadow seems to naturally prefer looking at correlators of $\{\Phi^{+}_{i\lambda}, \mathcal{O}^{+}_{i\lambda}\}$. i.e. single helicity combinations. One decouples in the soft limit, one acts like a current. Does this have an MHV, n-MHV, ... expansion flavor? Probably too soon to say based on what has been computed so far. In the following section, I will compute the two point correlators of $\{\Phi^{+}_{i\lambda}, \mathcal{O}^{+}_{i\lambda}\}$. With the smearing intrinsic to these modes, we are no longer restricted to contact-term structure. The computations will start from the Mellin amplitudes, however, if these Mellin+smeared combinations are what get promoted to primaries and they fit into some picture of dual operators, computations on the CFT side might be more natural in this basis and would go in the other direction, extracting the flat space amplitudes from these correlators.

IV. 2 PT FUNCTIONS

We will now compute the two point correlators of $\{\Phi^{+}_{i\lambda}, \mathcal{O}^{+}_{i\lambda}\}$ via known amplitudes at tree level. I will use the normalization

$$\langle\varphi^{+}_{i\lambda}(z,\bar{z})\varphi^{-}_{i\lambda'}(z',\bar{z}')\rangle = (2\pi)^4\frac{\pi\lambda}{\sinh\pi\lambda}\delta(\lambda+\lambda')\delta^2(z-z') \tag{IV.1}$$

coming from the plane wave Klein-Gordon norm and performing the Mellin transform, using the all-out helicity labels. The other $\varphi^{\pm}_{i\lambda}$ two point functions are zero. Note that shadows are normally used for completeness relations in computing CFT correlators. As such, an operator and its shadow produce a contact term. By picking the $\{\Phi^{+}_{i\lambda}, \mathcal{O}^{+}_{i\lambda}\}$ basis we have chosen a set that is independent of its shadows and yet still contains all of the information in the two helicity states. In this basis, all of the two point functions are of the form $\langle(\varphi^{+}_{i\lambda}(z,\bar{z}) \pm \tilde{\varphi}^{-}_{-i\lambda}(z,\bar{z}))(\varphi^{+}_{i\lambda}(z',\bar{z}') \pm \tilde{\varphi}^{-}_{-i\lambda}(z',\bar{z}'))\rangle$ and one can see that $\langle\varphi^{+}_{i\lambda}(z,\bar{z})\tilde{\varphi}^{-}_{i\lambda'}(z',\bar{z}')\rangle$ only appears with a single smear integral. The other terms have two φ of the same helicity which have zero two point function. We thus see that no contact terms arise in the two point function in this basis.

$$\begin{aligned}
\langle\mathcal{O}^{+}_{i\lambda}(z,\bar{z})\mathcal{O}^{+}_{i\lambda'}(z',\bar{z}')\rangle &= \langle\varphi^{+}_{i\lambda}(z,\bar{z})\tilde{\varphi}^{-}_{-i\lambda'}(z',\bar{z}')\rangle + \langle\tilde{\varphi}^{-}_{-i\lambda}(z,\bar{z})\varphi^{+}_{i\lambda'}(z',\bar{z}')\rangle \\
\langle\mathcal{O}^{+}_{i\lambda}(z,\bar{z})\Phi^{+}_{i\lambda'}(z',\bar{z}')\rangle &= i\left[-\langle\varphi^{+}_{i\lambda}(z,\bar{z})\tilde{\varphi}^{-}_{-i\lambda'}(z',\bar{z}')\rangle + \langle\tilde{\varphi}^{-}_{-i\lambda}(z,\bar{z})\varphi^{+}_{i\lambda'}(z',\bar{z}')\rangle\right] \\
\langle\Phi^{+}_{i\lambda}(z,\bar{z})\Phi^{+}_{i\lambda'}(z',\bar{z}')\rangle &= \langle\varphi^{+}_{i\lambda}(z,\bar{z})\tilde{\varphi}^{-}_{-i\lambda'}(z',\bar{z}')\rangle + \langle\tilde{\varphi}^{-}_{-i\lambda}(z,\bar{z})\varphi^{+}_{i\lambda'}(z',\bar{z}')\rangle
\end{aligned} \tag{IV.2}$$

Irrelevance of ordering coming from symmetry under $z \leftrightarrow z'$ and $\lambda \leftrightarrow \lambda'$ of each term (as seen by computing one) implies that each term appearing above is the same. So we see that not only are there no contact terms, butΦ and $\mathcal{O}$ are orthogonal:

$$\begin{aligned}
\langle\mathcal{O}^{+}_{i\lambda}(z,\bar{z})\mathcal{O}^{+}_{i\lambda'}(z',\bar{z}')\rangle &= \tfrac{2(2\pi)^4\Gamma[1+i\lambda]\Gamma[2+i\lambda]\delta(\lambda-\lambda')}{(z-z')^2|z-z'|^{2i\lambda}} \\
\langle\mathcal{O}^{+}_{i\lambda}(z,\bar{z})\Phi^{+}_{i\lambda'}(z',\bar{z}')\rangle &= 0 \\
\langle\Phi^{+}_{i\lambda}(z,\bar{z})\Phi^{+}_{i\lambda'}(z',\bar{z}')\rangle &= \tfrac{2(2\pi)^4\Gamma[1+i\lambda]\Gamma[2+i\lambda]\delta(\lambda-\lambda')}{(z-z')^2|z-z'|^{2i\lambda}}
\end{aligned} \tag{IV.3}$$

The way in which the quanitites raised to imaginary powers are always real here shows that z and $\bar{z}$ should rotate with opposite phases, helping us set requirements on analytical continuations that arise.

As a quick check let us see what would happen if we had chosen as our basis $\{\Phi^{+}_{i\lambda}, \mathcal{O}^{-}_{i\lambda}\}$ then the two point function $\langle\mathcal{O}^{-}_{i\lambda}(z,\bar{z})\Phi^{+}_{i\lambda'}(z',\bar{z}')\rangle$ can have a contact term. This is the only possible term, as seen by comparing the φ's that appear:

$$\langle\mathcal{O}^{-}_{i\lambda}(z,\bar{z})\Phi^{+}_{i\lambda'}(z',\bar{z}')\rangle = i\left[\langle\varphi^{-}_{i\lambda}(z,\bar{z})\varphi^{+}_{i\lambda'}(z',\bar{z}')\rangle - \langle\tilde{\varphi}^{+}_{-i\lambda}(z,\bar{z})\tilde{\varphi}^{-}_{-i\lambda'}(z',\bar{z}')\rangle\right]. \tag{IV.4}$$

We will use some integrals from [14] to see what happens to the contact terms here. The convolutions that arise meet the criteria of the appendix of [14], namely integrals like their (A.1):

$$\begin{gathered}
I_n = \tfrac{1}{2\pi}\int d^2z f_n(z)\bar{f}_n(\bar{z}) \\
f_n(z) = \prod_{i=1}^{n}\tfrac{1}{(z-z_i)^{h_i}}, \quad \bar{f}_n(\bar{z}) = \prod_{i=1}^{n}\tfrac{1}{(\bar{z}-\bar{z}_i)^{\bar{h}_i}}
\end{gathered} \tag{IV.5}$$

meeting the requirements of their (A.2):

$$\sum_{i=1}^{n} h_i = \sum_{i=1}^{n}\bar{h}_i = 2, h_i - \bar{h}_i \in \mathbb{Z} \tag{IV.6}$$

up to convergence requiring $h_i+\bar{h}_i < 2$. When we hit that bound we assume we can analytically continue and use their result. (Here $h_i + \bar{h}_i = 2 \pm 2i\lambda$ so perhaps special care for contributions at $\lambda = 0$, as I have seen appear in scalar computations, may arise upon further study of soft limits and appropriate basis choices.) Then, (A.3) and (A.4) will be sufficient to compute the two point correlators we are after:

$$I_2 = \frac{\Gamma[1-h_1]\Gamma[1-h_2]}{\Gamma[\bar{h}_1]\Gamma[\bar{h}_2]}(-1)^{h_1-\bar{h}_1}2\pi\delta^2(z_1-z_2). \qquad \text{(IV.7)}$$

This tells us the second term in IV.9 is proportional to

$$\begin{aligned}&\Gamma[2+i\lambda]\Gamma[2-i\lambda]\int d^2w\frac{1}{(w-z)^{i\lambda}(w-z')^{2-i\lambda}}\frac{1}{(\bar{w}-\bar{z})^{2+i\lambda}(\bar{w}-\bar{z}')^{-i\lambda}}\\&=\Gamma[1-i\lambda]\Gamma[1+i\lambda]\delta^2(z-z')\end{aligned} \qquad \text{(IV.8)}$$

since we used the result of [14] with $h_1 = i\lambda, h_2 = 2 - i\lambda, \bar{h}_1 = 2+i\lambda, \bar{h}_2 = -i\lambda$. The evaluation of this integral implies that, after plugging in the other terms that have been left out, the two terms in IV.9 actually cancel:

$$\langle \mathcal{O}^-_{i\lambda}(z,\bar{z})\Phi^+_{i\lambda'}(z',\bar{z}')\rangle \ = 0 \qquad \text{(IV.9)}$$

This is consistent with the above argument that any of the pairs $\{\Phi^+_{i\lambda},\mathcal{O}^+_{i\lambda}\}$, $\{\Phi^+_{i\lambda},\mathcal{O}^-_{i\lambda}\}$,$\{\Phi^-_{i\lambda},\mathcal{O}^+_{i\lambda}\}$,$\{\Phi^-_{i\lambda},\mathcal{O}^-_{i\lambda}\}$ form a natural basis of independent operators. We have just shown the orthogonality and lack of contact terms for two of the four sets, the other two just exchange helicities. We have thus reached the interesting conclusion:

There are no contact terms for the two point functions among the operators in this basis. The contact terms that one sees from the Mellin amplitudes alone are because those Mellin operators are a linear admixture of these basis operators and their shadows, which in working the other way (starting from a CFT) are typically designed just so as to produce these contact terms as part of completeness relations.

V. CONCLUSIONS

To smear or not to smear? Beyond the fact that shadow terms are allowed by $SL(2,\mathbb{C})$; appear in massless limits (when you don't try to throw terms away to get back to a Mellin transform that has 'nice' $\mathcal{CS}^2$ locality); and allow non-singular low point functions – the above computations indicate that some form of smearing is necessary to capture soft decoupling.

In effect, the soft/symmetry story which inspired a highest weight basis seems to push further towards incorporating Mellin+shadow terms. What I appreciate most is the following: once one decides to consider shadow terms, there appear to me three natural ways to fix the choice of linear combinations: soft decoupling; massless limits; and non-singular, orthogonal two point functions. Each of these independently determines a set of preferred coefficients. It is highly nontrivial that, together, they suggest a consistent basis that meets each of the three 'goals' simultaneously. i.e. something seems rather convincing in the manner in which the results of sections I, II, III, IV fit together. Moreover, a natural subset of operators emerges in a manner uncannily reminiscent of only-holomorphic currents and non-commuting double soft limits in non-abelian gauge theories. It is nice how the computations here combined somewhat disparate results from soft theorem analyses [6]; massive transforms [5]; and Mellin transforms [1, 4]. While an extremely speculative but nonetheless motivating arc of this recent highest weight basis study is to see if it is possible to connect amplitude recursion relations to CFT OPEs, it seems reassuring that, regardless, this type of pursuit may provide answers to some of the questions that have arisen in the soft/symmetry story that motivated this program in the first place. One roadblock to standard expansions of 4 pt functions in conformal blocks seemed to be the singular low point amplitudes (coming from 4D kinematics) and the shadow terms smear these out in a manner worth investigating further in future work.

ACKNOWLEDGEMENTS

I am grateful to A. Strominger and A. Zhibeodov for useful conversations as well as ongoing work with A. Strominger and S.H. Shao on pure-Mellin transforms [4]. This work was supported by the NSF and by the Hertz Foundation through a Harold and Ruth Newman Fellowship.

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