Classical Interpretation of the Weinberg Soft Factor

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I show how the radiation emitted during the scattering of non-relativistic charged particles corresponds to the $\mathcal{O}(\omega^{-1})$ soft factor in QED. Namely, if we think of the photon momentum in the soft factor as labeling a direction at which a far-field observer sits, the QED matrix element pre-factor corresponds to the time integral of the radiated electric field measured by that observer when a set of non-relativistic charged particles scatter and accelerate.

I. CLASSICAL SCATTERING

Consider the mode expansion of $F_{uz} = \partial_u A_z$ from "Low's Subleading Soft Theorem as a Symmetry of QED":

$$F_{uz} = -\frac{e\hat{\epsilon}^+_{\bar{z}}}{8\pi^2}\int_0^\infty d\omega\ \omega[a_+(\omega\hat{x})e^{-i\omega u} + a_-(\omega\hat{x})^\dagger e^{i\omega u}]. \qquad (I.1)$$

Its integral over u is given by:

$$\int du\ F_{uz} = -\frac{e\hat{\epsilon}^+_{\bar{z}}}{8\pi}\lim_{\omega\to 0^+}\omega[a_+(\omega\hat{x}) + a_-(\omega\hat{x})^\dagger] \qquad (I.2)$$

so that a soft insertion picks out: $\omega\hat{\epsilon}^+_{\bar{z}}$ times the Weinberg soft factor.

Semi-classically, we can think of the mode expansion of F_{uz} as the Fourier transform for the corresponding electric field component. The Weinberg soft theorem thus corresponds to the time integral of the radiated electric field measured at any far-field point labelled by $(z, \bar{z})$. Such a non-zero time-integrated value would be expected for a charged particle that accelerates.

Let's look at some equations from classical electrodynamics:

$$\vec{E}_{rad} = \vec{e}_r \times \left(\vec{e}_r \times \frac{\partial \vec{A}_{rad}}{\partial t}\right) \qquad (I.3)$$

where $\frac{\partial \vec{A}_{rad}}{\partial t}$ becomes the u derivative of the gauge field component tangent to the two sphere at the far-field point, just as F_{uz} is the u derivative of A_z. For a non-relativistic accelerating particle:

$$\vec{E}_{rad} = \frac{Q}{4\pi\epsilon_0 rc^2}\vec{e}_r \times (\vec{e}_r \times \vec{a}) \qquad (I.4)$$

so that the time integral of $\vec{E}_{rad}$ is proportional to the change in velocity of the particle. For instance, in the non-relativistic regime where the same particles come in and out, but with different velocities:

$$\int dt\ \vec{E}_{rad} = \sum_{in-out}\frac{Q_k}{4\pi\epsilon_0 rc^2}\vec{e}_r \times (\vec{e}_r \times \vec{v}_k). \qquad (I.5)$$

II. CONNECTION TO QED SOFT FACTOR

For a far-field point labeled by $(z, \bar{z})$, we have:

$$\vec{e}_r = \left(\frac{z+\bar{z}}{1+z\bar{z}}, \frac{i(\bar{z}-z)}{1+z\bar{z}}, \frac{1-z\bar{z}}{1+z\bar{z}}\right) \qquad (II.1)$$

while a particle traveling with four momentum:

$$p_k = |\mathbf{p}_k|\left(\sqrt{1+\frac{m_k^2}{|\mathbf{p}_k|^2}}, \frac{z_k+\bar{z}_k}{1+z_k\bar{z}_k}, \frac{i(\bar{z}_k-z_k)}{1+z_k\bar{z}_k}, \frac{1-z_k\bar{z}_k}{1+z_k\bar{z}_k}\right) \qquad (II.2)$$

has, at leading order in the non-relativistic limit:

$$\vec{v}_k = \frac{|\mathbf{p}_k|}{m_k}\left(\frac{z_k+\bar{z}_k}{1+z_k\bar{z}_k}, \frac{i(\bar{z}_k-z_k)}{1+z_k\bar{z}_k}, \frac{1-z_k\bar{z}_k}{1+z_k\bar{z}_k}\right). \qquad (II.3)$$

We then find that

$$\sum_{in-out}\left\{\frac{Q_k}{r}\vec{e}_r \times (\vec{e}_r \times \vec{v}_k)|\right\}\cdot\partial_z\vec{x} = \sum_{in-out} -2\frac{Q_k|\mathbf{p}_k|}{m_k r}\frac{(\bar{z}_k-\bar{z})(1+z_k\bar{z})}{(1+z\bar{z})^2(1+z_k\bar{z}_k)} \qquad (II.4)$$

where $\vec{x} = r\vec{e}_r$.

Meanwhile, in the low-particle-momentum limit

$$\sum_{in-out}\omega\hat{\epsilon}^+_{\bar{z}}\frac{p_k\cdot\epsilon^+}{p_k\cdot q} = \sum_{in-out} -2\frac{Q_k|\mathbf{p}_k|}{m_k r}\frac{(\bar{z}_k-\bar{z})(1+z_k\bar{z})}{(1+z\bar{z})^2(1+z_k\bar{z}_k)} \qquad (II.5)$$

for photon momentum and polarization four vectors given by:

$$q = \omega\left(1, \tfrac{z+\bar{z}}{1+z\bar{z}}, \tfrac{i(\bar{z}-z)}{1+z\bar{z}}, \tfrac{1-z\bar{z}}{1+z\bar{z}}\right)$$
$$\epsilon^+ = \tfrac{1}{\sqrt{2}}(\bar{z}, 1, -i, -\bar{z}). \qquad (II.6)$$

Similarly,

$$\sum_{in-out}\left\{\frac{Q_k}{r}\vec{e}_r \times (\vec{e}_r \times \vec{v}_k)\right\}\cdot\partial_{\bar{z}}\vec{x} = \sum_{in-out}\omega\hat{\epsilon}^+_{\bar{z}}\frac{p_k\cdot\epsilon^-}{p_k\cdot q}. \qquad (II.7)$$

where $\epsilon^-_\mu = \epsilon^{+*}_\mu$ in Minkowski coordinates.

We thus see that the Weinberg soft factor appearing in the mode expansion of F_{uz} in the $\omega \to 0$ limit corresponds to the total time integral of the electric field radiating towards $(z, \bar{z})$ coming from the acceleration of massive charged particles when their velocities change in a scattering process.

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