Generalizing the Soft Factor/Classical Connection

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Here, I generalize the connection between soft factors and classical solutions described in "Classical Interpretation of the Weinberg Soft Factor," showing that the same relation also holds when the scattered charged particles are massless. Namely: the polarization vector times the soft factor for a photon emitted in the $\hat{x}$ direction during a scattering process is proportional to the time integral of the classical radiated electric field for the same process, as measured by an observer sitting along that direction at a large distance from the scattering process. While this is the opposite velocity limit, compared to the previous paper, the underlying connection is the same: the duality between position space and momentum space for the photon field near null infinity. The low frequency limit picks out the large distance, classical behavior, which in turn aligns the position and momentum space directions.

I. SOFT FACTOR AS EXPECTATION VALUE

Again, I start with the mode expansion for $F_{uz} = \partial_u A_z$ from "Low's Subleading Soft Theorem as a Symmetry of QED":

$$F_{uz} = -\frac{e\hat{\epsilon}^{+}_{\bar{z}}}{8\pi^2}\int_0^\infty d\omega\ \omega[a^{\text{out}}_{+}(\omega\hat{x})e^{-i\omega u} + a^{\text{out}}_{-}(\omega\hat{x})^\dagger e^{i\omega u}]. \quad \text{(I.1)}$$

where the integral over u is given by:

$$\int du\ F_{uz} = -\frac{e\hat{\epsilon}^{+}_{\bar{z}}}{8\pi}\lim_{\omega\to 0^+}\omega[a^{\text{out}}_{+}(\omega\hat{x}) + a^{\text{out}}_{-}(\omega\hat{x})^\dagger]. \quad \text{(I.2)}$$

Here, I have made the fact that these operators correspond to outgoing photons explicit, relating to the fact that I am measuring the radiation in a far-field region a long time after the scattering process generating it occurred.

Let's draw some intuition from quantum mechanics. If you have an operator $\mathcal{O}$ and a state $|\psi\rangle$ the expectation value of the operator in this quantum state is:

$$\frac{\langle\psi|\mathcal{O}|\psi\rangle}{\langle\psi|\psi\rangle} \quad \text{(I.3)}$$

In some sense, it is then natural to say that if I have a scattering process going from some $|\text{in}\rangle$ state to some $|\text{out}\rangle$ state, I can think of:

$$\frac{\langle\text{out}| : \mathcal{O}\mathcal{S} : |\text{in}\rangle}{\langle\text{out}|\mathcal{S}|\text{in}\rangle} \quad \text{(I.4)}$$

like an expectation value of the operator for that process. Here, the denominator is the matrix element describing the transition amplitude between the $|\text{in}\rangle$ and $|\text{out}\rangle$ states. As opposed to the expectation value given a fixed state in (I.3), both the numerator and denominator in (I.4) are transition amplitudes. Time ordering the operator $\mathcal{O}$ with the scattering matrix $\mathcal{S}$ is used to explicitly distinguish operators which modify the incoming and outgoing states.

Now let $\mathcal{O} = \int du F_{uz}(r\to\infty,\hat{x})$, where I have made the remaining spatial dependence of (I.2) explicit. This corresponds to the long-distance radiated electric field measured at some point labeled by the direction $\hat{x}$ on the S^2 at infinity.

As in (I.4), I would like to interpret the soft part that factorizes from the matrix element:

$$\begin{aligned}-\tfrac{e\hat{\epsilon}^{+}_{\bar{z}}}{8\pi}&\lim_{\omega\to 0^+}\omega\langle\text{out}|[a^{\text{out}}_{+}(\omega\hat{x}) + a^{\text{out}}_{-}(\omega\hat{x})^\dagger]\mathcal{S}|\text{in}\rangle\\ &= -\tfrac{e\hat{\epsilon}^{+}_{\bar{z}}}{8\pi}\lim_{\omega\to 0^+}\omega S^{(0)+}\langle\text{out}|\mathcal{S}|\text{in}\rangle\end{aligned} \quad \text{(I.5)}$$

as the classical expectation value of the radiated electric field integrated over time.

From a QFT point of view, the zero frequency limit has extracted the Weinberg pole in the soft factor for the matrix element describing the scattering process of $|\text{in}\rangle$ to $|\text{out} + 1\text{ soft photon @ }\hat{x}\rangle$. Note here, that this $\hat{x}$ is the direction of the soft photon in momentum space. The key to the connection between the classical measurement and the QFT soft factor is that the massless photon localizes in the large r limit to the same point on the position space sphere as its direction in momentum space. The multiplication by ω not only picks out just the Weinberg pole in the soft factorization but, by leaving just the $\hat{x}$ dependence, also allows me to use the (I.4) notion of an expectation value for operator (I.2) to arrive at the position-space interpretation of the soft factor as a classical measurement of the time integral of the radiated electric field at large r, made by an observer sitting at $\hat{x}$ on a far away sphere.

In the particular gauge choice:

$$A_r = 0, \qquad A_u = \mathcal{O}(r^{-1}) \quad \text{(I.6)}$$

only the fields A_z and $A_{\bar{z}}$ remain in the large r limit, so that $F_{uz} = \partial_u A_z$ and (I.2) takes the form $A^+_z - A^-_z$, the difference of the gauge field at the future and past u limits of $\mathcal{I}^+$, future null infinity.[1] By going to this gauge, I can express an observable field strength in terms of boundary values of the gauge field.

[1] arXiv:1407.3789

II. MASSLESS SCATTERING

In "Classical Interpretation of the Weinberg Soft Factor," I showed that in the limit of non-relativistic scattering of charged particles, this connection between soft factors and classical measurements held, using results from classical electromagnetism. The essential reason why the interpretation worked is that the classical observable I was interested in depended only on the charges and momenta of the incoming and outgoing particles, which are the same variables used to define the $|\text{in}\rangle$ and $|\text{out}\rangle$ states. Here, I use the gauge field solution for a set of massless charges emerging from the spacetime origin[2] and show that the classical value is again the soft factor.

The LSET gauge choice is equivalent to $A_r = 0$ in our null coordinates ($t = u + r$). After using conservation of charge, the radial dependence drops out and:

$$A_\mu(u, r, \hat{x}) = (-\sum_j Q_j \log\left[-\frac{p_j \cdot n}{E_j}\right] \delta(u), 0, 0, 0) \quad \text{(II.1)}$$

where n is a radial null vector parameterized by the direction $\hat{x}$. Using a $(u, z, \bar{z})$ dependent gauge transformation, I can convert this expression into the further constrained gauge choice (I.6). The result is that $A_u = A_r = 0$ while:

$$A_i = \partial_i \sum_j Q_j \log\left[-\frac{p_j \cdot n}{E_j}\right] \theta(u) \quad \text{(II.2)}$$

for $i \in \{z, \bar{z}\}$. This solution has the nice property that is an exact 1-form on the S^2 where the u dependence implies that the value jumps when the wavefront of the massless particles passes the observer's position. Taking the large r limit does not affect the numerical form of the expression since all of the particles are moving on the same light shell, however, this limit has the nice interpretation of allowing me to superpose massless scattering processes starting at different points, where the finite shifts in origin give subleading effects.

Using (II.2), the classical value of the operator (I.2) becomes:

$$\begin{aligned} A_z^+ - A_z^- &= \partial_z \sum_j Q_j \log\left[-\tfrac{p_j \cdot n}{E_j}\right] \\ &= \sum_j Q_j \tfrac{p_j \cdot \partial_z n}{p_j \cdot n} \end{aligned} \quad \text{(II.3)}$$

At the same time, the soft factor gives:

$$\begin{aligned} \hat{\epsilon}_{\bar{z}}^+ \lim_{\omega \to 0^+} \omega S^{(0)+} &= \sum_j Q_j \hat{\epsilon}_{\bar{z}}^+ \tfrac{p_j \cdot \epsilon^+}{p_j \cdot \tilde{n}} \\ &= \sum_j Q_j \hat{\epsilon}_z^{+*} \tfrac{p_j \cdot \epsilon^+}{p_j \cdot \tilde{n}} \\ &= \partial_z \tilde{n} \cdot \sum_{j,\alpha} Q_j \epsilon^{\alpha *} \tfrac{p_j \cdot \epsilon^\alpha}{p_j \cdot \tilde{n}} \\ &= \sum_j Q_j \tfrac{p_j \cdot \partial_z \tilde{n}}{p_j \cdot \tilde{n}} \end{aligned} \quad \text{(II.4)}$$

where $\tilde{n} = \frac{q}{\omega}$. Here, I have dropped pre-factors corresponding to different normalization conventions for the gauge fields. The third line uses the fact that $\partial_z \tilde{n}(\hat{x}) \cdot \epsilon^{-*}(\hat{x}) = 0$ so that the completeness relation for polarization vectors and charge conservation, $\sum Q_j = 0$, can be used to arrive at the fourth line.

The final results of (II.3) and (II.4) are the same when we set $\tilde{n}$ corresponding to the direction of the soft photon, equal to n corresponding to the direction at which the classical field is measured. What is interesting is that, while from the QFT interpretation the soft factor naturally gets exponentiated to allow multiple soft insertions, there is less of a motivation to do so for the time integrated quantity in the classical interpretation, except for computing correlation functions, for example. Since it can be interpreted as a closed 1-form on the S^2, $A^+ - A^- = d\phi$, correlation functions of the scalar ϕ rather than A at different points, are more natural, and would be the analog of multiple soft emissions in different directions.

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[2] arXiv:1401.7667