Massive Soft Factors

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I show that the soft factor corresponds to the measured time integrated radiation for any speed.

I. MASSIVE CHARGES

Start with the Liénard-Wiechert radiation field for a single accelerating charge:

$$\vec{E}_{rad}(r,t) = \frac{Q}{4\pi\epsilon_0 rc}\left[\frac{\vec{e}_r \times ((\vec{e}_r - \vec{\beta}(t')) \times \dot{\vec{\beta}}(t'))}{(1-\vec{e}_r\cdot\vec{\beta}(t'))^3}\right] \quad \text{(I.1)}$$

where $\vec{\beta} = \frac{\vec{v}}{c}$ and t' is the delayed time:

$$ct' = ct - r + \vec{e}_r\cdot\vec{s}(t') = u + \vec{e}_r\cdot\vec{s}(t') \quad \text{(I.2)}$$

for a source at position $\vec{s}(t')$. In the following, I will take units where $c = 1$ and suppress the $4\pi\epsilon_0$ normalization. From (I.2), one finds:

$$\frac{dt}{dt'} = 1 - \vec{e}_r\cdot\vec{\beta}(t'). \quad \text{(I.3)}$$

I will now consider a superposition of (I.1) for a set of massive charges accelerating from zero velocity to $\vec{p}_k$ such that the accelerations are always parallel to the velocities. In this case:

$$\vec{\beta}_k(t') = \frac{\vec{p}_k}{E_k} f_k(t') \quad \text{(I.4)}$$

for some functions $f_k(t')$ that go from 0 to 1 over the time during which the particles accelerate.

An observer sitting at $R\vec{e}_r$ for some fixed, large R will then observe the following time-integrated radiation field:

$$\begin{aligned}
&\partial_i\vec{x}\cdot\int dt\vec{E}_{rad}\\
&= \sum_k \int (1-\vec{e}_r\cdot\vec{\beta}_k(t'))dt' Q_k\left[\frac{\vec{e}_r\times(\vec{e}_r\times\dot{\vec{\beta}}_k(t'))}{(1-\vec{e}_r\cdot\vec{\beta}_k(t'))^3}\right]\cdot\partial_i\hat{x}\\
&= \sum_k \int dt' Q_k\left[\frac{\vec{e}_r\times(\vec{e}_r\times\frac{\vec{p}_k}{E_k}\dot{f}_k(t'))}{(1-\vec{e}_r\cdot\frac{\vec{p}_k}{E_k}f_k(t'))^2}\right]\cdot\partial_i\hat{x}\\
&= \sum_k \int_0^1 df_k Q_k\left[\frac{\vec{e}_r\times(\vec{e}_r\times\frac{\vec{p}_k}{E_k})}{(1-\vec{e}_r\cdot\frac{\vec{p}_k}{E_k}f_k)^2}\right]\cdot\partial_i\hat{x}\\
&= \sum_k Q_k\partial_i\log\left[-\frac{p_k\cdot n}{E_k}\right]
\end{aligned} \quad \text{(I.5)}$$

where the first equality uses the fact that $\vec{\beta}\times\dot{\vec{\beta}} = 0$ from (I.4), and (I.2) to change the integration measure from dt to dt'. This cancels a factor of $(1-\vec{e}_r\cdot\vec{\beta}_k(t'))$ in the denominator, as seen in the second equality. In the third equality, the t' integral is converted to an integral over f_k, which evaluates to the final equality.

II. INTERPRETATION

The last line in (I.5) agrees with the results in "Classical Interpretation of the Weinberg Soft Factor" and "Generalizing the Soft Factor/Classical Connection," in which the two opposite limits of 1) non-relativistic and 2) massless charged particles, were considered. Using charge conservation, this was seen as the Weinberg soft factor with the position of the far away observer n replacing the direction of the soft photon momentum:

$$\begin{aligned}\int du F_{uz} &\propto \sum_k Q_k\partial_z\log\left[-\tfrac{p_k\cdot n}{E_k}\right]\\ &= \hat{\epsilon}_z^{+*}\omega S^{(0)+}.\end{aligned} \quad \text{(II.1)}$$

Note that changing the integration interval for f_i from $[0,1]$ to $[1,0]$ is equivalent to having the particle decelerate. This changes the sign of the i^{th} particle's contribution to (I.5), just as the sign of its contribution to the soft factor would switch.

Here, I have shown that the result discussed in the previous papers for particular velocity limits holds for charged particles of any mass or velocity, when they are forced to accelerate on linear trajectories. In the QFT picture, one treats the interactions as occurring in a small space-time region. In this regime, $\vec{s}$ remains small, while $\vec{\beta}$ need not be. One can then take the limit in which the accelerations are instantaneous. In this case, the soft factor appears as a classical background pulse of radiation emitted from the interaction point, as relevant to Section II of "Subtleties of Zero Modes."

ACKNOWLEDGEMENTS

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[1] "Classical Interpretation of the Weinberg Soft Factor"
[2] "Generalizing the Soft Factor/Classical Connection"
[3] "Subtleties of Zero Modes"