An Alternate Approach for Finding an Orthogonal Basis

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I present an alternative to the Gram-Schmidt method for finding a basis of orthogonal vectors spanning the same space as a set of starting vectors.

The Gram-Schmidt method for creating an orthogonal basis that spans a set of vectors is as follows: 1) Start with one of the vectors and divide by its norm to get a unit vector. 2) Take another vector, subtract off its projection along the first vector and normalize the result. 3) Repeat for each subsequent vector in the set, subtracting off projections along all previous unit vectors that have been found.

The alternative that I present is elegant although computationally slower because it involves calculating multiple determinants. It is advantageous in situations where unknown symbolic variables need to be manipulated, or where small numbers cause instabilities using Gram-Schmidt. My determinant approach provides an orthogonal basis without requiring division, although scaling by the modulus of the vectors can be done at the end if a normalized basis is desired.

My alternative also uses linear algebra, but starts from determinants rather than projections of vectors. The determinant of a set of linearly dependent vectors is zero. When a matrix has a non-zero determinant, the rows can be taken as a basis for the space spanned by the set of vectors.

Begin by considering the cross product from vector calculus. The cross product of two vectors can be written as a determinant with arbitrary unit vectors occupying the first row:

$$\vec{A}\times\vec{B}=\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{1}$$

The result is a vector that is orthogonal to $\vec{A}$ and $\vec{B}$, as can be seen by the following route: $\vec{C}\cdot(\vec{A}\times\vec{B})$ is equivalent to replacing the first row of the determinant in Equation 1 by $\vec{C}$. Since $\vec{C}=\vec{A}\times\vec{B}$, it follows that $\vec{C}\cdot\vec{C}$ will be non-zero as long as $\vec{C}\neq\vec{0}$. This is the case as long as $\vec{A}$ and $\vec{B}$ are not parallel. As a result, the determinant of:

$$M=\begin{pmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{pmatrix} \tag{2}$$

is non-zero and the three rows are linearly independent.

Moreover, $\vec{C}$ will be perpendicular to both $\vec{A}$ and $\vec{B}$. However, $\vec{A}$ and $\vec{B}$ need not be perpendicular to one another. If I replace the original $\vec{B}$ that I used to find $\vec{C}$ in Equation 1 with the vector $\vec{C}$, I find a new vector that is orthogonal to both $\vec{A}$ and $\vec{C}$:

$$\vec{D}=\vec{A}\times\vec{C}=\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ C_x & C_y & C_z \end{vmatrix} \tag{3}$$

Since $\vec{C}$ is already orthogonal to $\vec{A}$, I have just found an orthogonal basis for three dimensions using two determinants, without needing to do any division.

This can be extended to higher dimensions in a way that: 1) gives a basis for the full space spanned by a set of m linearly independent m-dimensional vectors; and 2) includes a known subset of these basis vectors which span the same n-dimensional space as a set of n linearly independent vectors of length m which are pxrovided as a starting point. This second feature is important when one wants to study that particular subspace.

Consider the case with $n<m$. I require that the n starting vectors of length m be linearly independent and, additionally, that the vectors found by truncating to the first $n+1$ components are also linearly independent. Truncate these vectors to get n vectors of length $n+1$ and construct an $n\times n$ determinant similar to Equation 1, with arbitrary unit vectors in the first row:

$$\vec{v}_{n+1}=\begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 & \ldots \\ v_1^{x_1} & v_1^{x_2} & v_1^{x_3} & \ldots \\ v_2^{x_1} & v_2^{x_2} & v_2^{x_3} & \ldots \\ . & . & . & \end{vmatrix} \tag{4}$$

This determinant will give an $n+1$ dimensional vector orthogonal to the previous n. Next, include the $n+2^{th}$ coordinate of the first n vectors and add a zero as the $n+2^{th}$ coordinate of $\vec{v}_{n+1}$ from Equation 4. Place this vector in the last row of an $n+2\times n+2$ determinant and repeat.

This will build up to a set of m vectors, of which $\vec{v}_{n+1}...\vec{v}_m$ will be orthogonal to each other and the starting set $\vec{v}_1...\vec{v}_n$. One can now work backwards, using steps analogous to Equation 3 to sequentially replace the first n vectors with ones orthogonal to all other vectors, using $m\times m$ determinants.