Conformal Mapping of Displacement Vectors

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I explore the asymmetry between displacements in space and time for motion in one dimension.

The existence of a maximum velocity c establishes a notion of symmetry between space and time: it gives a way to equate distances to time intervals such that a space-time diagram can appear to put ct and x on equal footing. There exists a fundamental difference between displacements in space and time, however. While I can move either forward or backward in x, I can only move forward in time. If I take my current position as the origin of my coordinate system, the region of possible displacements $(c\mathrm{d}t, \mathrm{d}x)$ only fills a half-plane.

This raises a question about whether $\mathrm{d}x$ and $c\mathrm{d}t$ are the best coordinates to use to describe a displacement. For instance, using a polar coordinate system to describe a change in position in the (x, y) plane (where my current position is always taken as the origin), I could describe my motion as a series of positive $\mathrm{d}r$ displacements along different directions θ that parameterize the slope. While such a system may be convenient for a single, instantaneous displacement, it makes reconstructing the full path more challenging than if $(\mathrm{d}x, \mathrm{d}y)$ were given. It is ideal to find descriptions of displacements that are independent of the observer.

In what follows, I consider what happens when I map the half plane $(c\mathrm{d}t, \mathrm{d}x)$ onto a full plane. It is possible to perform a conformal mapping of this type by temporarily treating the displacement as a complex variable $z = c\mathrm{d}t + i\mathrm{d}x$ and squaring to get $w = \frac{z^2}{2}$. The real and imaginary parts are then used to define the horizontal and vertical coordinates in this new plane: $\mathrm{d}s^2 = \frac{1}{2}(c^2\mathrm{d}t^2 - \mathrm{d}x^2) = \mathrm{d}\eta\mathrm{d}\xi$ as the horizontal coordinate and $c\mathrm{d}t\mathrm{d}x$ as the vertical coordinate.

The result of this mapping is depicted in Figure 1. It stretches angles at the origin by a factor of 2, but preserves the orthogonality of lines of constant $c\mathrm{d}t$ and $\mathrm{d}x$. The curved lines in the $(c\mathrm{d}t, \mathrm{d}x)$ plane show contours of constant $\mathrm{d}s^2$ and $c\mathrm{d}t\mathrm{d}x$, while the curved lines in the $(\mathrm{d}s^2, c\mathrm{d}t\mathrm{d}x)$ plane show contours of constant $c\mathrm{d}t$ and $\mathrm{d}x$.

Because this map squares the magnitude of the initial space-time displacement, the coordinates of the map are now area elements. What is intriguing is that the two orthogonal area element coordinates correspond to those of the Minkowski (ct, x) diagram and the rotated (η, ξ) diagram of my paper "Motivating Special Relativity using Linear Algebra." While the former (ct, x) basis diagonalizes the metric, the latter (η, ξ) basis diagonalizes the Lorentz transformation.

This mapping also highlights the $x \to -x$ symmetry that was important in my linear algebra derivation: it moves $(0, -\mathrm{d}x)$ and $(0, +\mathrm{d}x)$ to the same point on the negative $\mathrm{d}\eta\mathrm{d}\xi$ axis. An irreconcilable ambiguity in the definition of the map along this axis would arise if we could not consider these displacements as fundamentally the same in some respect. Restricting $v < c$ is equivalent to requiring displacement vectors to have a positive value of $\mathrm{d}\eta\mathrm{d}\xi$. Placing the same restriction in the $(c\mathrm{d}t, \mathrm{d}x)$ plane would be analogous to saying that a particle could only move forward in time if there was no maximum velocity.

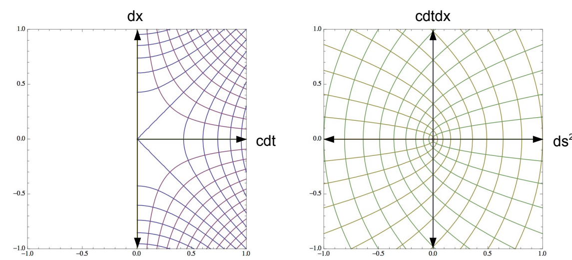

FIG. 1. Conformal mapping of space-time displacements.

In my linear algebra derivation of Special Relativity, I showed that the area element $\mathrm{d}\eta\mathrm{d}\xi$ is invariant under boosts. Meanwhile $c\mathrm{d}t$ depends on the frame. In this new coordinate system, the displacements along the horizontal axis behave similar to time displacements in Galilean transformations: they are the same regardless of the reference frame. As such, we can divide by $|\mathrm{d}s|$ to get a set of displacements, rather than area elements, that can be more easily compared to Galilean intuition.

For a particle traveling at a constant positive β, the time-like and space-like displacements are:

$$\mathrm{d}\tau = \sqrt{1-\beta^2}\mathrm{d}t \qquad \mathrm{d}\chi = \frac{1}{\sqrt{1-\beta^2}}\mathrm{d}x. \tag{1}$$

These coordinates describe the increment in proper time, τ, and the length contraction of distances parallel to the motion.

The mapping from half-plane to full plane thus illustrates important features of Special Relativity. Moreove, these features of the $w(z)$ mapping are consistent with Special Relativity because the scale factor for the time axis relative to the space axis, c, was equal to the speed of light.