Dot Products in Special Relativity

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I present a derivation of the Minkowski metric and visualizations of the space-time dot product.

I. POSTULATES

1. The dot product is linear and commutative.
2. The dot product is invariant under boosts.
3. The metric defining the dot product is independent of the reference frame.

II. DERIVING THE METRIC

The goal is to construct a scalar from two space-time vectors that is invariant during Lorentz boosts between inertial reference frames. Requiring that the dot product be linear (Postulate 1) gives:

$$\begin{aligned} A\cdot B &= (a_t \;\; a_x)\begin{pmatrix} \hat{t}_a\cdot\hat{t}_b & \hat{t}_a\cdot\hat{x}_b \\ \hat{x}_a\cdot\hat{t}_b & \hat{x}_a\cdot\hat{x}_b \end{pmatrix}\begin{pmatrix} b_t \\ b_x \end{pmatrix} \\ &= A'MB \end{aligned} \tag{1}$$

where the matrix M corresponds to the Minkowski metric, written $\eta_{\mu\nu}$ for Special Relativity. I use M to distinguish it from the (η,ξ) basis defined in my "Motivating Special Relativity Using Linear Algebra" paper.

If the dot product is assumed to be commutative: $A\cdot B = B\cdot A$, so M is symmetric. This holds in any basis. Moreover, the (ct,x) basis should diagonalize M so that a displacement in the rest frame is orthogonal to a change in time. This intuition will be used as a cross check. Starting in the (η,ξ) frame:

$$M = \begin{pmatrix} f & g \\ g & h \end{pmatrix}. \tag{2}$$

I have used the symmetry of M and a restriction that the entries are real to reduce M to three parameters, corresponding to dot products of the $\hat{\eta}$ and $\hat{\xi}$ unit vectors as in Equation 1 for the (ct,x) basis.

As shown in the linear algebra-based derivation paper, a boost stretches η by $\lambda_1(v)$ and ξ by $\lambda_2(v)$. This stretching can be absorbed into M':

$$M' = \begin{pmatrix} \lambda_1^2 f & \lambda_1\lambda_2 g \\ \lambda_1\lambda_2 g & \lambda_2^2 h \end{pmatrix} = M. \tag{3}$$

The matrices M and M' can be equated by choosing A and B to pick out each entry, satisfying Postulate 2.

Postulate 3 says that M should be independent of β. While $M = M'$ would hold if $f \propto 1/\lambda_1^2$, the definition of the dot product should not change with reference frames. This is satisfied if $f = h = 0$. Moreover, $\lambda_1\lambda_2 = 1$ gives $M = g\sigma_x$.

Finally, the dot product should reduce to the ordinary dot product for two vectors along the $\hat{x} = \frac{\hat{\xi}-\hat{\eta}}{\sqrt{2}}$ axis. This gives $g = -1$ for the convention where spatial dot products are positive. Rotating back into the (ct,x) basis:

$$M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

I have thus derived the space-time metric for a single spatial coordinate, finding the invariant dot product between two space-time vectors: $A\cdot B = -a_t b_t + a_x b_x$.

III. VISUALIZING DOT PRODUCTS

While spatial bases can be rotated onto each other, for boosts, the transformations are restricted: time-like and space-like vectors on opposite sides of the light cone cannot be transformed into one another.

The above derivation generalizes to three spatial dimensions by having $\hat{y}$ and $\hat{z}$ behave like $\hat{x}$ so that: $A\cdot B = -a_t b_t + \vec{a}\cdot\vec{b}$, where $\vec{a}$ is the spatial part of A. It is possible to rotate the spatial axes so that $\hat{x}$ aligns with A and then only use (b_t, b_x).

To build intuition, consider the case $A = B$. This dot product is zero if $a_t = a_x$ and is largest if one of the two components is zero. For general B: $|A\cdot B| = |(a_x, a_t, 0)\times(b_t, b_x, 0)|$, illustrating the cross-product-like nature of the space-time dot product when time is treated as an additional spatial coordinate. The magnitude of this cross product is equal to the area defined by reflecting one of the vectors across the line $x = ct$. Alternatively, a standard dot product can be taken after reflecting one of the vectors in time. Both methods are illustrated in Figure 1.

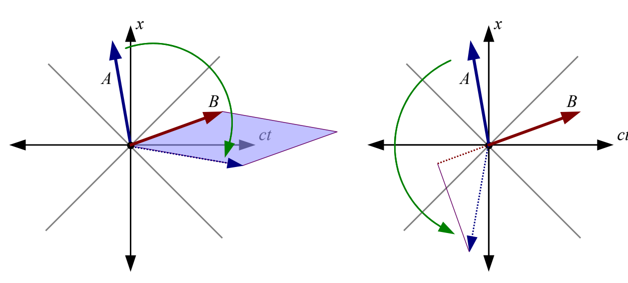

FIG. 1. Two space-time dot product visualizations which use reflections in the (ct,x) plane.