Position as a Single-Valued Function of Time

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I present a geometrical illustration of how the speed limit in Special Relativity requires position to be a single-valued function of time in any reference frame.

I. BACKGROUND

In my paper "Motivating Special Relativity using Linear Algebra," I derived the Lorentz transformations assuming that the speed of light is the same in any reference frame and that transformations of space-time coordinates are linear. In the (η, ξ) basis where:

$$\eta = \frac{ct - x}{\sqrt{2}} \quad \xi = \frac{ct + x}{\sqrt{2}} \tag{1}$$

a boost in velocity is described by:

$$\begin{pmatrix} \eta' \\ \xi' \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix}$$
$$\lambda_1 = \sqrt{\tfrac{1+\beta}{1-\beta}}, \quad \lambda_2 = \sqrt{\tfrac{1-\beta}{1+\beta}}. \tag{2}$$

The fact that $|\beta| > 1$ makes λ_1 and λ_2 imaginary justifies the notion of a speed limit in Special Relativity: $|v| \leq c$. In this paper, I consider the implications of having the strict inequality $|v| < c$ hold.

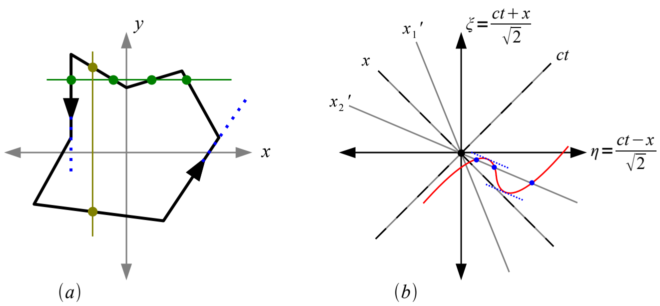

FIG. 1. Slope restrictions lead to properties of $x(ct)$.

II. SPACE-TIME TRAJECTORIES

A function $f(k)$ of a variable k is defined such that for any value of that variable k, which f takes as an input, the output $f(k)$ has a single value. For instance, $f(k) = k^2$ has a single value for each k, while two different values of k are allowed to give the same value of $f(k)$, e.x. k and $-k$.

The motion of a particle is described by a path in the (ct, x) plane. If we can write the position as a function of ct, then while a particle can be at the same position at different times, at any time it can only have a single position.

While daily experience makes this property of x and ct seem intuitive, such a restriction need not hold for any two physical quantities. Figure 1a illustrates how the relationship between a particle's x and y position does not have this restriction. In this case, the slope $\frac{dy}{dx}$ represents a direction, which only takes on a physical meaning if there is something that distinguishes that direction, e.x. the amount of sunlight hitting you if you move in or out of a shadow by traveling that way. Indeed, if one were unable to distinguish x from y, we could rotate our coordinates to arbitrarily change the slope.

I will show that setting a limit on the magnitude of $\frac{dx}{dt}$ implies that x is a single-valued function of t, which we can write as $x(ct)$. Figure 1b shows that the transformations of Equation 2 can modify the slope of the position axis in different reference frames. Since:

$$\frac{\lambda_2}{\lambda_1} = \frac{1 - \beta}{1 + \beta} > 0 \tag{3}$$

for $|\beta| < 1$, and the x axis has slope -1 in the rest frame, lines of constant time for any reference frame will have a negative slope when in plotted the (η, ξ) coordinates of a given rest frame.

The restriction that $|\beta| < 1$ also implies that the the trajectory of a particle in (η, ξ) is strictly increasing as a function of η. If its slope were zero at any point, the particle would be traveling at c in the $-\hat{x}$ direction. If it's slope were infinite, the particle would be traveling at c in the $+\hat{x}$ direction. At any instant, the slope must be between these two values.

I will show that this implies that $x = x(ct)$ for any reference frame by contradiction (the red curve in Figure 1b). If there exists a reference frame in which x' takes on more than one value for a given ct' then for a trajectory that is continuous (the particle does not jump between points in space and time) there will be some position along the particle's path in the (η, ξ) plane that has a tangent parallel to the average slope, which is the slope of the x' axis. This comes from an application of the mean value theorem of calculus.

I previously showed that the slope of any x' axis is negative, so this means that the slope of the trajectory in (η, ξ) would also be negative, which is not allowed by $|\beta| < 1$. This contradiction tells us that x is single-valued as a function of time in any reference frame. Special Relativity thus implies that the particle cannot occupy two positions at the same time.