# Motivating Special Relativity using Linear Algebra

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I present a geometrical derivation of results from Special Relativity for a single spatial coordinate.

## I. POSTULATES

In this paper, I derive results from Special Relativity using symmetry and the following postulates:

1. The speed of light is constant in any reference frame.
2. The transformation of space-time coordinates when changing between reference frames is linear.

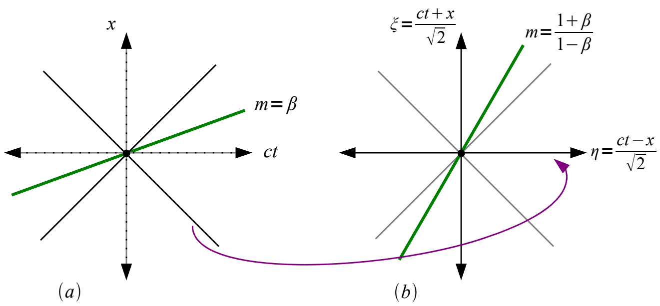


FIG. 1. Choice of basis to diagonalize the transformation.

## II. TRANSFORMING COORDINATES

When describing the motion of a particle along the $x$ axis, it is convenient to plot position as a function of time. Figure 1a shows such a plot. The green line has slope $\beta = \frac{v}{c}$ and describes the motion of a particle which moves at a constant velocity $v$ and passes through $x = 0$ at $t = 0$. The two diagonals describe the paths of photons traveling at speed $c$ in the $\pm\hat{x}$ directions.

According to Postulate 1, if we change the velocity of our reference frame by "boosting" along the $x$ axis, the speed of light will still be $c$, meaning that in the new reference frame, the paths of photons will still have slope $\pm 1$. If, as per Postulate 2, the transformation is linear, these diagonals will be eigenvectors of the transformation. Figure 1b rotates the $(ct, x)$ coordinates by 45° to the $(\eta, \xi)$ basis that diagonalizes the boost. In this basis, a boost in velocity amounts to applying the linear transformation:

$$\begin{pmatrix} \eta' \\ \xi' \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix} \tag{1}$$

for some eigenvalues $\lambda_1(v)$ and $\lambda_2(v)$ which determine the scaling of the axes during a boost by $v$.

In this $(\eta, \xi)$ basis, the slope of a particle traveling at a constant velocity is $m = \frac{1+\beta}{1-\beta}$. A boost into a frame in which this particle has zero velocity must take this slope to 1:

$$\frac{\Delta\xi'}{\Delta\eta'} = \frac{\lambda_2}{\lambda_1}\frac{\Delta\xi}{\Delta\eta} = \frac{\lambda_2}{\lambda_1} m = 1 \quad \rightarrow \quad \frac{\lambda_2}{\lambda_1} = \frac{1-\beta}{1+\beta}. \tag{2}$$

Now consider a switch in the sign of $v$ by adding a second particle moving in the opposite direction. If both the $+v$ and $-v$ particles pass through $x = 0$ at $t = 0$, their positions at any time will be reflections of one another across the $m = 1$ diagonal in the $(\eta, \xi)$ plane, which corresponds to the time axis. If scaling $\eta$ by $\lambda_1(v)$ and $\xi$ by $\lambda_2(v)$ brings the $+v$ particle's space-time coordinate at a given $ct$ to a particular $ct'$ on the $m = 1$ diagonal during a $+v$ boost, then scaling $\eta$ by $\lambda_2(v)$ and $\xi$ by $\lambda_1(v)$ will bring the corresponding space-time coordinate of a $-v$ particle's path to the same point on the $m = 1$ diagonal. Since this is equivalent to performing a $-v$ boost instead, $\lambda_1(-v) = \lambda_2(v)$.

Because $\lambda_1(v)$ determines the scaling of the $\eta$ axis during a boost by $v$, a subsequent boost by $-v$ should undo this rescaling, giving $\lambda_1(-v) = 1/\lambda_1(v)$. This says that $\lambda_1(v)\lambda_2(v) = 1$: area elements are invariant under a boost. Solving for $\lambda_1$ and $\lambda_2$ gives:

$$\lambda_1 = \sqrt{\frac{1+\beta}{1-\beta}} \quad \lambda_2 = \sqrt{\frac{1-\beta}{1+\beta}} \tag{3}$$

where the limit of $\lambda \to 1$ for $v \to 0$ sets the overall sign of the eigenvalues.

From the definition of $\eta$ and $\xi$:

$$\eta = \frac{ct - x}{\sqrt{2}} \quad \xi = \frac{ct + x}{\sqrt{2}}, \tag{4}$$

this transformation reduces to:

$$\begin{aligned} ct' &= \gamma(ct - \beta x) \\ x' &= \gamma(x - \beta ct) \end{aligned} \tag{5}$$

in the $(ct, x)$ basis, with $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. This completes a derivation of the standard Lorentz transformation for a single spatial coordinate in Special Relativity.

## III. APPLICATIONS

Figure 2 illustrates the effect of a boost and gives a geometrical picture from which the velocity addition formula, length contraction, time dilation, the invariant interval, and the relativistic Doppler effect will be derived.

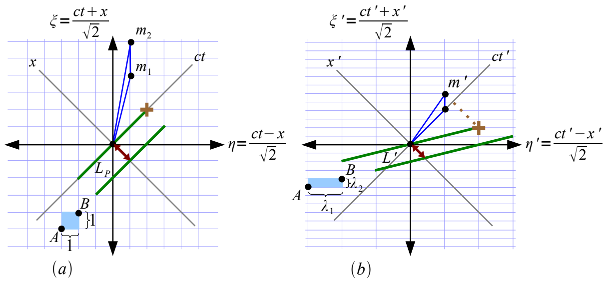


FIG. 2. Illustration of a boost in the $(\eta, \xi)$ basis.

### III.1. Velocity Addition

If one particle is traveling at $v_1$, another at $v_2$, the relative speed as seen from the reference frame of particle 1 will not generally be $v_2 - v_1$. To get the correct result, take a triangle with one vertex at the origin, one at the point $(1, m_1)$, and one at the point $(1, m_2)$, as illustrated by the blue lines in Figure 2a. Next, boost to a frame where $m_1$ is along the diagonal, corresponding to the rest frame of particle 1 (Figure 2b).

In this frame, the point $(1, m_2)$ transforms to:

$$\sqrt{\tfrac{1+\beta_1}{1-\beta_1}} \begin{pmatrix} 1 \\ m' \end{pmatrix} = \sqrt{\tfrac{1+\beta_1}{1-\beta_1}} \begin{pmatrix} 1 \\ \frac{1-\beta_1}{1+\beta_1} m_2 \end{pmatrix}. \tag{6}$$

Solving for $m'$ gives:

$$\beta' = \frac{\beta_2 - \beta_1}{1 - \beta_1 \beta_2} \tag{7}$$

as the relative velocity. The velocity addition formula from Special Relativity for two particles moving along the same axis follows from taking $\beta_1 \to -\beta_1$.

### III.2. Length Contraction

Figure 2 also illustrates length contraction. A solid object at rest traces out a diagonal ribbon parallel to the slope $m = 1$ time axis. Let one edge be at $x = 0$ and the other be at $x = -L_P$. This gives $\xi = \eta$ and $\xi = \eta - \sqrt{2} L_P$, corresponding to the top and bottom green lines in Figure 2a, respectively. When the reference frame is boosted by $v$ in Figure 2b, the edges are described by:

$$\xi' = \frac{1-\beta}{1+\beta}\eta' \quad \& \quad \xi' = \frac{1-\beta}{1+\beta}\eta' - \sqrt{\frac{1-\beta}{1+\beta}} \times \sqrt{2} L_P. \tag{8}$$

Physical length is measured at constant time. This is shown by the red arrows, which mark the separation, as measured along the $x$ and $x'$ axes, between the top and bottom green lines in the stationary and boosted frames. The line $\xi' = -\eta'$ in Figure 2b intersects the top green line at $(0, 0)$ and the bottom green line at $\sqrt{1-\beta^2} \times \frac{L_P}{\sqrt{2}}(1, -1)$. The distance between these points is $\sqrt{1-\beta^2} L_P$. The length of the object in the moving frame is thus contracted: $L' = \frac{L_P}{\gamma}$, compared to the proper length $L_P$ in the object's rest frame.

### III.3. Time Dilation

Proper time is the time between two events at the same $x$. This corresponds to the distance between the origin and the brown cross in Figure 2a. The points $(0, 0)$ and $\frac{ct_P}{\sqrt{2}}(1, 1)$ transform to:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \& \quad \frac{ct_P}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{1+\beta}{1-\beta}} \\ \sqrt{\frac{1-\beta}{1+\beta}} \end{pmatrix} \tag{9}$$

in Figure 2b. Here, the time separation is the distance between the $x'$ axis and the dashed brown line:

$$ct' = \frac{\eta' + \xi'}{\sqrt{2}} = \frac{ct_P}{\sqrt{1-\beta^2}} \quad \rightarrow \quad t' = \gamma t_P. \tag{10}$$

This result is known as time dilation. The time between two events is longer in a reference frame where those events occur at different $x$ positions.

### III.4. The Invariant Interval

The fact that area elements are invariant tells us that:

$$\begin{aligned} d\eta \times d\xi &= \tfrac{1}{2}(cdt - dx) \times (cdt + dx) \\ &\propto dx^2 - c^2 dt^2 \end{aligned} \tag{11}$$

is invariant under boosts. Plotting the coordinates of two events, $A$ and $B$, in the $(\eta, \xi)$ plane, the area of a rectangular envelope with these two events at opposite corners (the light blue regions in Figure 2) is proportional to the invariant interval between these events: $\Delta s^2 = \Delta x^2 - c^2 \Delta t^2$.

### III.5. Relativistic Doppler Effect

In the $(\eta, \xi)$ plane, horizontal lines correspond to photons traveling in the $-\hat{x}$ direction, while vertical lines correspond to photons traveling in the $+\hat{x}$ direction. The frequency observed by a person at $x = 0$ is inversely proportional to the distance between intersections of these gridlines and the time axis (the $m = 1$ diagonals shown in gray in Figure 2).

Since a boost in the $+\hat{x}$ direction stretches $\eta$ by $\sqrt{\frac{1+\beta}{1-\beta}}$, the vertical gridlines in the boosted frame are further apart than in the rest frame. The frequency of light moving in the $+\hat{x}$ direction is thus redshifted by a factor of $\sqrt{\frac{1-\beta}{1+\beta}}$ as the observer moves away from the source.

Since the horizontal gridlines are closer together in the boosted frame, the frequency of light moving in the $-\hat{x}$ direction is blue shifted by a factor of $\sqrt{\frac{1+\beta}{1-\beta}}$.