Visualization for Spin-1/2 Inner Products

Sabrina Gonzalez Pasterski

(Dated: July 23, 2013)

I present a geometrical visualization for the magnitude of the inner product of two spin-1/2 states.

A spin-1/2 quantum state can be visualized as a vector on the Bloch sphere, pointing in a direction described by the angles (θ, ϕ) in spherical coordinates. These angular coordinates parameterize the quantum spin state. For instance, the states $|\psi\rangle$ and $|\chi\rangle$ corresponding to spins pointing in the $\vec{v}_\psi$ and $\vec{v}_\chi$ directions:

$$\begin{aligned}\vec{v}_\psi &= (\sin\theta_1\cos\phi_1, \sin\theta_1\sin\phi_1, \cos\theta_1)\\ \vec{v}_\chi &= (\sin\theta_2\cos\phi_2, \sin\theta_2\sin\phi_2, \cos\theta_2)\end{aligned} \tag{1}$$

are represented in quantum mechanics by:

$$\begin{aligned}|\psi\rangle &= \cos\tfrac{\theta_1}{2}|0\rangle + e^{i\phi_1}\sin\tfrac{\theta_1}{2}|1\rangle\\ |\chi\rangle &= \cos\tfrac{\theta_2}{2}|0\rangle + e^{i\phi_2}\sin\tfrac{\theta_2}{2}|1\rangle\end{aligned} \tag{2}$$

where the two orthogonal basis states $|0\rangle$ and $|1\rangle$ correspond to spin up and spin down along the $\hat{z}$ direction.

One aspect that makes visualizing these states counter-intuitive is that a spin polarized along the $+\hat{z}$ direction is orthogonal to a spin polarized along the $-\hat{z}$ direction. In physical space, the dot product of these two unit vectors would be -1, not zero.

When we visualize light passing through a polarizer, we find that tilting a filter by 45° cuts the intensity of an initially polarized beam in half, tilting by 90° cuts it out completely, while flipping by a full 180° has no effect. For spin-1/2 particles, such as electrons, the magnetic moment can be oriented either up $(+\frac{\hbar}{2})$ or down $(-\frac{\hbar}{2})$ along a particular direction. Unlike rotating a light polarizer by 90°, measuring a $+\hat{z}$ spin along a perpendicular direction gives spin up and down with equal probability.

This arises from the fact that the inner product between two states behaves differently than the dot product between the vectors in Equation 1:

$$\begin{aligned}|\langle\chi|\psi\rangle|^2 = &\left|\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + e^{i(\phi_1-\phi_2)}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right|^2\\ = &\left(\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \cos(\phi_1-\phi_2)\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right)^2\\ &+\left(\sin(\phi_1-\phi_2)\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right)^2\\ = &\frac{1}{2}\left[1+\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)+\cos\theta_1\cos\theta_2\right].\end{aligned} \tag{3}$$

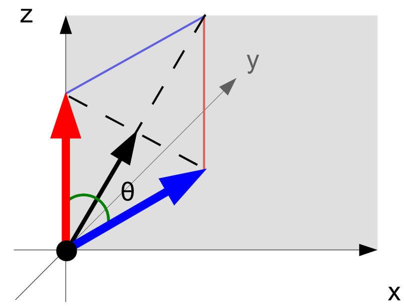

FIG. 1. Illustration of the calculation in Equation 4.

In this paper I derive a method to obtain the same value using a geometrical visualization where vectors behave as they normally would in physical space. Take the norm squared of the average of the two spin vectors:

$$\begin{aligned}\left|\frac{\vec{v}_\psi+\vec{v}_\chi}{2}\right|^2 = &\frac{1}{4}[(\sin\theta_1\cos\phi_1+\sin\theta_2\cos\phi_2)^2\\ &+(\sin\theta_1\sin\phi_1+\sin\theta_2\sin\phi_2)^2\\ &+(\cos\theta_1+\cos\theta_2)^2]\\ = &\frac{1}{2}\left[1+\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)+\cos\theta_1\cos\theta_2\right]\end{aligned} \tag{4}$$

The results of Equations 3 and 4 are equal. As a check, we see that they both give zero for two vectors pointing in opposite directions on the Bloch sphere. The algebra can be simplified by taking the case where one spin points along $+\hat{z}$, as illustrated in Figure 1.

In quantum mechanics the norm squared of an inner product corresponds to a transition probability. The visualization presented here connects this notion of probability to what one would see in a classical double slit interference experiment, where superimposed electric field vectors are added and then squared to get the intensity.

Here, the vectors are the spin orientations of two spin-1/2 states, and the transition probability corresponds to the likelihood of achieving a spin up measurement along the $\vec{v}_\chi$ direction for an electron with a spin in the $\vec{v}_\psi$ direction.