Mellin Transform of MHV 4 Point

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This note computes the highest-weight transform of the color-ordered four point tree level MHV amplitude.

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I. CONVENTIONS

The main result of [1] was an expression for the scattering amplitude of $SL(2,\mathbb{C})$ highest-weight states in terms of an integral transform acting on the $\mathcal{S}$-matrix elements in the standard plane wave basis:

$$\boxed{\tilde{\mathcal{A}}_{\Delta_1,\cdots,\Delta_n}(w_i,\bar{w}_i) \equiv \prod_{i=1}^{n}\int_0^\infty \frac{dy_i}{y_i^3}\int dz_i d\bar{z}_i\, G_{\Delta_i}(y_i,z_i,\bar{z}_i;w_i,\bar{w}_i)\,\mathcal{A}(m_j\hat{p}_j^\mu)\,.}$$

Here G_Δ is a bulk-to boundary propagator of weight Δ, with $(y,z,\bar{z})$ Poincaré coordinates on the H_3 parameterizing the on-shell momentum space hyperboloid of the asymptotic states

$$\hat{p}^\mu(y,z,\bar{z}) = \left(\frac{1+y^2+|z|^2}{2y}, \frac{\mathrm{Re}(z)}{y}, \frac{\mathrm{Im}(z)}{y}, \frac{1-y^2-|z|^2}{2y}\right)\,, \tag{I.1}$$

and $(w,\bar{w})$ the conformal S^2 coordinates of a reference null direction

$$q^\mu = \left(1+|w|^2, w+\bar{w}, -i(w-\bar{w}), 1-|w|^2\right)\,. \tag{I.2}$$

In these coordinates G_Δ then takes the form:

$$G_\Delta(y,z,\bar{z};w,\bar{w}) = \left(\frac{y}{y^2+|z-w|^2}\right)^\Delta\,. \tag{I.3}$$

I.1. Massless Limit

In terms of plane waves, the highest weight states being scattered in $\tilde{\mathcal{A}}$ can be written as:

$$\phi^\pm_{\Delta,m}(X^\mu;w,\bar{w}) = \int_0^\infty \frac{dy}{y^3}\int dz d\bar{z}\, G_\Delta(y,z,\bar{z};w,\bar{w})\,\exp\left[\pm im\,\hat{p}^\mu(y,z,\bar{z})\,X_\mu\right]\,.$$

In the $y\to 0$ limit G_Δ behaves as:

$$G_\Delta(y,z,\bar{z};w,\bar{w}) \sim Cy^{2-\Delta}\delta^2(z-w) + \frac{y^\Delta}{|z-w|^{2\Delta}} + \ldots \tag{I.4}$$

where the constant C can be computed by matching the area integral on the S^2 as shown in the appendix of [2]. As in [2] and the appendix of [1] the weight will take values on a contour $\Delta = a + i\lambda$. Note that while the real power of the mass scaling is the same for both terms when $a=1$, the imaginary parts of the exponent differ (ie $2-\Delta=\bar{\Delta}$).

The next step is to do the momentum space convolution. We take $\omega \equiv \frac{m}{2y}$ fixed in the $y\to 0$ limit. Then the contribution of the first term in the y expansion to $\phi^\pm_{\Delta,m\to 0}$ would be

$$\begin{aligned}\phi^\pm_{\Delta,m}(X^\mu;w,\bar{w}) &\propto \int_0^\infty dy y^{-1-\Delta}\int dz d\bar{z}\delta^2(z-w)\exp\left[\pm i\omega\, q^\mu(z,\bar{z})\,X_\mu\right] + \ldots\\ &= \left(\frac{2}{m}\right)^\Delta \int_0^\infty d\omega\omega^{\Delta-1}\exp\left[\pm i\omega\, q^\mu(w,\bar{w})\,X_\mu\right] + \ldots\end{aligned} \tag{I.5}$$

where we have used the $m\to 0$ limit to reduce p to a massless four momentum and also to motivate doing the integral of the expansion in small y, since here we are considering it as an expansion in small m with ω fixed. The second term will produce a similar contribution with a power of $m^{\Delta-2}$. Here we will assume $\mathrm{Re}[\Delta]$ is slightly greater than one to motivate picking the first term in the $y\to 0$ expansion, after an overall mass-dependent rescaling, as the limit of the massive transform for $m=0$ states.

The ϕ profiles resulting from convolving with either term in (I.4) are highest-weight solutions. The first term has the advantage of being local on the S^2; ie the reference direction with respect to which the state is highest weight is the same as the direction of the null momentum of the particle. Meanwhile the remaining integral of the transform in the radial direction on the momentum space hyperboloid is converted in this limit to a Mellin transform of the energy. Once we have motivated the Mellin transformed state in this manner, we will take the limit where the real part of the contour returns to $a=1$, which will be convenient for formal manipulations to express the distributional nature of the dependence on the imaginary parts of the weights.

For example acting directly on the plane wave creation and annhiliation operators to define:

$$\mathfrak{a}_\lambda(\hat{q}) \equiv \int_0^\infty d\omega\;\omega^{i\lambda}a(\omega,\hat{q}) \tag{I.6}$$

gives an inner product

$$\langle 0|\mathfrak{a}_{\lambda'}(\hat{q}')\mathfrak{a}^\dagger_\lambda(\hat{q})|0\rangle = 2(2\pi)^4\delta(\lambda-\lambda')\delta^{(2)}(\hat{q}-\hat{q}') \tag{I.7}$$

after using the change of variables $u=\ln\omega$ to write

$$\begin{aligned}\int_0^\infty \frac{d\omega}{\omega^{1+ib}} &= \frac{1}{2\pi}\int_0^\infty \frac{d\omega}{\omega}e^{-ib\ln\omega}\\ &= \int_{-\infty}^\infty du e^{-ibu}\\ &= 2\pi\delta(b)\end{aligned} \tag{I.8}$$

which relies on the fact that ω is real and positive and in particular on the real part of the power of ω appearing in the integrand.

Note that (I.6) matches the highest weight transform for Re$[\Delta] = 1$. The Mellin transformed states are orthogonal for different weights and directions. Through this transform we have exchanged energies for weights while the direction on the sphere is the same as the direction of the four momenta [4] (see footnote 9 of [5] for an extended-BMS motivation for scattering Rindler energy eigenstates).

II. COMPUTATION

Note that the Mellin transformed states are those appearing in [2] and in [3]. With the perspective of [1] we will be transforming the amplitude directly rather than foliating Minkowski space (see section IV for a comment on the connection between the foliation perspective and our transform point of view). Noticing (I.5) is equivalent to a weighted integral over all energies for particle momenta parallel to a reference null four vector, we can define the massless analog of the transform in [1], converting from plane wave to highest weight scattering amplitudes as:

$$\tilde{\mathcal{A}}_{\Delta_1,\cdots,\Delta_n}(w_i,\bar{w}_i) \equiv \prod_{k=1}^{n}\int_0^\infty d\omega_k \omega_k^{i\lambda_k}\,\mathcal{A}(\omega_k q_k^\mu)\,. \tag{II.1}$$

where the particle momenta on the right hand side have the form $p_k^\mu = \omega_k q^\mu(w_k,\bar{w}_k)$ for q as in (I.2). This amplitude is a multiple Mellin transform of the plane wave one. Note the Mellin transforms are of the energies not the Mandelstam invariants.

For the scalar case, such a convolution gave highest weight states of weight $1+i\lambda$. We will see that the energy dependence of the stripped amplitude naturally shifts the real parts of the weights of the fields appearing in the highest weight amplitude in a helicity-dependent manner as is also suggested by footnote 9 of [5].

We will now look at the structure for the tree level color-ordered n-point MHV amplitude. The Parke-Taylor formula for the delta-function stripped tree level amplitude gives:

$$A = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle} \tag{II.2}$$

for all $+$ except for i,j $-$ helicity in the all-out convention. In our convention for coordinates on the celestial sphere:

$$\langle ij\rangle = 2\sqrt{\omega_i\omega_j}(w_i - w_j) \tag{II.3}$$

thus the energy dependence factors out of the delta-function stripped amplitude and plays the role of shifting the weight of the highest-weight states that appear in the Mellin transform, in accord with the dimension of the field. Meanwhile the holomorphic vs antiholomorphic spinors that would appear in the anti-MHV amplitudes give the spin dependence of CFT-like correlators on the celestial S^2, as consistent from the 4D scattering perspective, with little group scaling.

Before specializing to the 4 point case, we point out one feature present in all MHV tree level amplitudes. If we start to compute the transformed amplitude

$$\tilde{\mathcal{A}} = \prod_{k=1}^{n}\int_0^\infty d\omega_k \omega_k^{i\lambda_k}\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}\delta^4(\sum_k p_k) \tag{II.4}$$

by first factoring out the overall scale of the energies (ie fix $\sum \omega_k = s$) we are left with

$$\tilde{\mathcal{A}} \propto \int_0^\infty ds s^{i\sum\lambda_k - 1} = 2\pi\delta(\sum \lambda_k) \tag{II.5}$$

where the measure from the simplex integral contributes dss^{n-1}, the Parke-Taylor formula gives an overall energy scaling of s^{4-n}, the four dimensional delta function gives s^{-4}, and the Mellin transform integrand gives $s^{i\sum\lambda_k}$. Combining these contributions we see from (I.8) that the transformed amplitude, for any number of scatterers n, has delta function support on the locus in weight space where the sum of the imaginary parts is zero.

We will now consider $2 \to 2$ scattering in the crossed configuration with Parke-Taylor amplitude

$$A = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle} \tag{II.6}$$

the goal is to compute

$$\begin{aligned}\tilde{\mathcal{A}} = & -\frac{w_{12}^3}{w_{23}w_{34}w_{14}}\int\prod_k d\omega_k \omega_1^{1-i\lambda_1}\omega_2^{1-i\lambda_2}\omega_3^{-1+i\lambda_3}\omega_4^{-1+i\lambda_4} \\ & \times\delta^{(4)}(\omega_1 q_1 + \omega_2 q_2 - \omega_3 q_3 - \omega_4 q_4)\end{aligned} \tag{II.7}$$

Here we use conjugate $i\lambda$ weights for in versus outgoing states so the support will be on $\lambda_1+\lambda_2-\lambda_3-\lambda_4 = 0$. We make the choice $\omega_k = s\sigma_k$ where $\sum\sigma_k = 2$, as convenient for $2 \to 2$ scattering.

Shifting coordinates so that $w_i = z_i - z_2$, again for convenience, and writing the argument of the delta function in terms of the $q(w,\bar{w})$ of (I.2) one finds as an intermediate result the following expression

$$\begin{aligned}\tilde{\mathcal{A}} = & -\pi\delta(\lambda_1+\lambda_2-\lambda_3-\lambda_4)\frac{z_{12}^3}{z_{23}z_{34}z_{14}} \\ & \times\int_0^1 d\sigma_1\int_0^1 d\sigma_3 \sigma_1^{1-i\lambda_1}(1-\sigma_1)^{1-i\lambda_2}\sigma_3^{-1+i\lambda_3}(1-\sigma_3)^{-1+i\lambda_4} \\ & \times\delta^{(2)}(\sigma_1 z_{12} - \sigma_3 z_{34} + z_{24}) \\ & \times\delta(\sigma_1|z_{12}|^2 + \sigma_3\left[|z_{24}|^2 - |z_{23}|^2\right] - |z_{24}|^2)\end{aligned} \tag{II.8}$$

where we have used the $p_{tot}^0 + p_{tot}^3$ constraint combined with $\sum\sigma_k = 2$ to see that $\sigma_1+\sigma_2 = \sigma_3+\sigma_4 = 1$. The p_{tot}^1, p_{tot}^2 delta function has support on the solution:

$$\sigma_1 = \frac{z_{23}\bar{z}_{24} - \bar{z}_{23}z_{24}}{z_{34}\bar{z}_{12} - \bar{z}_{34}z_{12}} \tag{II.9}$$

$$\sigma_3 = \frac{z_{24}\bar{z}_{12} - \bar{z}_{24}z_{12}}{z_{34}\bar{z}_{12} - \bar{z}_{34}z_{12}} \tag{II.10}$$

One must check that these are saturated within the range of integration for σ_k, which places constraints on the positions. Alternatively, one could take the point of view that whatever functional form of the answer we get will then be evaluated at points such that these conditions are met, which ultimately reduce to allowed kinematic configurations for $2 \to 2$ massless scattering. The Jacobian from this delta function gives a factor of $\frac{i}{2(z_{12}\bar{z}_{34}-\bar{z}_{12}z_{34})}$ where we will suppress a potential sign that comes from evaluating the absolute value.

At these values of σ_k the argument of the remaining delta function can be written as

$$\sigma_1|z_{12}|^2 + \sigma_3\left[|z_{24}|^2 - |z_{23}|^2\right] - |z_{24}|^2 = \frac{i\det\{q_1,q_2,q_3,q_4\}}{4(z_{12}\bar{z}_{34} - \bar{z}_{12}z_{34})} \quad \text{(II.11)}$$

where the determinant is of a matrix with each row one of the reference null momenta of the form (I.2) for each scatter. The $i(z_{12}\bar{z}_{34}-\bar{z}_{12}z_{34})$ dependent Jacobian is thus seen to cancel between the two delta functions leaving us at this point with an expression of the form:

$$\begin{aligned}\tilde{\mathcal{A}} = \; & -(2\pi)\delta(\lambda_1+\lambda_2-\lambda_3-\lambda_4)\frac{z_{12}^3}{z_{23}z_{34}z_{14}} \\ & \times\sigma_1^{1-i\lambda_1}(1-\sigma_1)^{1-i\lambda_2}\sigma_3^{-1+i\lambda_3}(1-\sigma_3)^{-1+i\lambda_4} \\ & \times\delta(\det\{q_1,q_2,q_3,q_4\})\end{aligned} \quad \text{(II.12)}$$

where σ_1 and σ_3 are evaluated at (II.9) and (II.10).

At this stage it is beneficial to reflect on the role the two remaining delta functions play. In order to have an allowed $2 \to 2$ scattering configuration in flat space the signed four momenta must sum to zero. This means that the reference vectors must form the edges of a closed null polygon in $\mathbb{R}^4$. Thus, these four vectors are not linearly independent. The vanishing of the determinant appearing here is precisely that statement. In the Mellin transform, we are performing a weighted integral over the moduli of possible lengths for four null vectors pointing in assigned directions to form a closed polygon. Now because of the dilation symmetry of the property of whether or not a polygon closes, once any such solution exists for given ratio of edge lengths, any uniform rescaling of the edges will also result in an allowed closed null polygon. This is the content of the first delta function where, by the change of variables in (I.8) and the choice of contour for the complex weights, we get a delta function for the signed sum of the imaginary parts of the weights from the dilation mode integral as shown in (II.5) to appear in any tree level MHV amplitude.

With these delta functions appearing in the transformed amplitude, it is natural to formally consider scattering smeared over the imaginary weights and positions on the celestial sphere, in a manner such that the delta functions then restrict the overlap profiles to the kinematically allowed configurations.

We will now write the second delta function in terms of conformal cross ratios. Using the conventions of Di Francesco et al (5.27) for the variable naming we let

$$\eta = \frac{z_{12}z_{34}}{z_{13}z_{24}} \quad \text{(II.13)}$$

in terms of which

$$\det\{q_1,q_2,q_3,q_4\} = 8|z_{13}|^2|z_{24}|^2\mathrm{Im}[\eta]. \quad \text{(II.14)}$$

While $|z_{13}|^2|z_{24}|^2$ can go to zero, this only occurs at singular configurations of the z_i. In the following we will consider the amplitudes of interest to be those smeared near the support of the locus $\mathrm{Im}[\eta]=0$ ie the single cross ratio in the 2D conformal theory is real.

We now want to simplify

$$\begin{aligned}\tilde{\mathcal{A}} = \; & -\frac{\pi}{4}\delta(\mathrm{Im}[\eta])\delta(\lambda_1+\lambda_2-\lambda_3-\lambda_4)\times\frac{1}{|z_{13}|^2|z_{24}|^2}\frac{z_{12}^3}{z_{23}z_{34}z_{14}} \\ & \times\sigma_1^{1-i\lambda_1}(1-\sigma_1)^{1-i\lambda_2}\sigma_3^{-1+i\lambda_3}(1-\sigma_3)^{-1+i\lambda_4}\end{aligned} \quad \text{(II.15)}$$

into the form of a standard 4 point function with weight dependent factors of z_{ij} and $\bar{z}_{ij}$ times some function of η which we see has support only for real η. This restriction to real η will be used in this simplification as it gives us relations like

$$\frac{z_{12}\bar{z}_{24}}{\bar{z}_{12}z_{24}} = \frac{z_{13}\bar{z}_{34}}{\bar{z}_{13}z_{34}} \quad \text{(II.16)}$$

from $\eta = \bar{\eta}$. After such simplifications we find

$$\boxed{\begin{aligned}\tilde{\mathcal{A}} = \; & (-1)^{1-i\lambda_2+i\lambda_3}\frac{\pi}{4}\left[\frac{\eta^5}{1-\eta}\right]^{1/3}\delta(\mathrm{Im}[\eta]) \\ & \times\delta(\lambda_1+\lambda_2-\lambda_3-\lambda_4)\prod_{i<j}^{4} z_{ij}^{h/3-h_i-h_j}\bar{z}_{ij}^{\bar{h}/3-\bar{h}_i-\bar{h}_j}\end{aligned}} \quad \text{(II.17)}$$

where the holomorphic and antiholomorphic weights are given by:

$$\begin{array}{ll} h_1 = -\frac{i}{2}\lambda_1 & \bar{h}_1 = 1-\frac{i}{2}\lambda_1 \\ h_2 = -\frac{i}{2}\lambda_2 & \bar{h}_2 = 1-\frac{i}{2}\lambda_2 \\ h_3 = 1+\frac{i}{2}\lambda_3 & \bar{h}_3 = \frac{i}{2}\lambda_3 \\ h_4 = 1+\frac{i}{2}\lambda_4 & \bar{h}_4 = \frac{i}{2}\lambda_4 \end{array} \quad \text{(II.18)}$$

as consistent with footnote 9 of [5], and $h = h_1+h_2+h_3+h_4$, $\bar{h} = \bar{h}_1+\bar{h}_2+\bar{h}_3+\bar{h}_4$.

III. CONCLUSIONS

In this note we have computed the highest-weight-transformed amplitude for the particular example of a 4 point tree level color-ordered $2 \to 2$ MHV scattering process.

Here, as in [1], the space of allowed kinematic configurations for on-shell 4D scattering plays a crucial role in these small-n computations, where Lorentz invariance – as manifested by global $SL(2,\mathbb{C})$ transformations of the coordinates on the celestial sphere – dictate the CFT-correlator-like structure.

The motivation for transforming amplitudes to a highest weight basis comes from recent work by Strominger et al demonstrating that the asymptotic symmetry group of gravity in asymptotically flat spacetimes can be extended to include superrotations [5] whose Ward identities correspond to a Virasoro symmetry on the celestial sphere. Recently a stress tensor for this effective 2D CFT in terms of the boundary metric at null infinity was proposed [6]. The hope is that what amounts to manipulations of Lorentz invariance in this note and [1] will become a much richer story once one couples to gravity.

It is interesting to note that the transformations which lead to these highest weight amplitudes convolve plane wave amplitudes at all energies. While at first it may seem like a detriment, since the scattering amplitudes computable from a low energy effective theory would be insufficient, there is a curious notion that if we were able to one day work out a holographic description of scattering in asymptotically flat spacetimes in terms of a 2D CFT, the CFT description would then tell us some properties of the behavior of the amplitudes at high energies we might otherwise be unable to probe.

IV. COMMENTS

The following are a few side comments related to what has been described in this note. It is worthwhile to compare the spacetime foliation and $\mathcal{S}$-matrix transform approaches to highest weight scattering. Since the transform depends on the amplitude at all energies, a crucial part of it is the momentum conserving delta function. If we write

$$(2\pi)^4\delta^4(\sum p_k) = \int d^4x e^{ix\cdot\sum p_k} \tag{IV.1}$$

the Mellin transform will act on both the stripped part of the amplitude and this factor, where in the simplest cases – like MHV amplitudes – the effect of the stripped amplitude is to shift the weight. Then since highest weight states have the form

$$\phi^{\pm}_{\Delta,m=0}(X^\mu; w, \bar{w}) = \frac{1}{(-X^\mu q_\mu \mp i\epsilon)^\Delta}\,. \tag{IV.2}$$

the foliation of Minkowski spacetime in, for instance, the forward lightcone of the origin turns the integral over the H_3 length scale (ie $x^2 = -\ell^2$) into something that factors out of what then looks like a convolution of bulk to boundary propagators on a position-space slice (ie $d^3x_{H_3}$ from the plane wave expansion of the momentum conserving delta function). This is just one trick for working backwards from the momentum space amplitude to a position space expression. More conventionally, one would start by just defining the highest weight scattering profiles in position space and compute the amplitude for scattering between them directly.

It may be interesting down the road to consider the effect of the second term in (I.4), since it is curious how for a contour at $\mathrm{Re}[\Delta] = 1$ both terms appear with the same real part for the power of mass. And while the first term is convenient for localizing integrals on the S^2 the smearing associated with second one will allow a CFT 2-point function overlap (like that shown for the massive case in the Appendix to [1]) in addition to the contact term in the Klein-Gordon norm. The contact term corresponds to the fact that a free massless particle will enter and leave traveling in the same direction while the second term would smear this profile so that these alternate highest weight states overlap at separated points as well, with the expected analytic behavior on the S^2 coming from global conformal symmetry arguments. The tradeoff is one between having simpler mode orthogonality relations and amplitude transform expressions in the Mellin case, versus being able to avoiding trivial results at low point amplitudes due to the kinematic restrictions for massless scattering when there is no such smearing. Leaving m small but nonzero would avoid the issue of not being able to scale out the same power of the mass in both terms, however if one takes this route they should be wary of the order of limits in taking m small to motivate picking out the small y behavior of G_Δ and then integrating this behavior over y, using the small mass to suppress the integrals of other terms in the y expansion.

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