# 4pt in 2+1

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This note considers the four point function of four massless fields in a 2+1 dimensional Massive-massless-massless cubic theory. I show that much of the work in 3+1 carries over straightforwardly (as if restricting to the celestial circle as far as the kinematics is concerned). The feature allowing a conformal block decomposition is that the four point function in 3 dimensions in a non-scale invariant theory does not over-constrain the Mellin transform. Unitarity in 2+1 follows from the change of basis for a theory that is unitary in momentum space, while the conformal block decomposition should be used to reveal features of the dual picture.

## I. FROM 3+1 TO 2+1

In 2+1 dimensions a null momentum can be parameterized by

$$p^\mu = \omega(1+z^2, 2z, 1-z^2) \tag{I.1}$$

where $z \in \mathbb{R}$. For four massless external momenta, the delta function coming from momentum conservation has the form:

$$\delta^{(3)}(p_1+p_2-p_3-p_4) = \frac{1}{4|z_{14}z_{13}z_{34}|}\delta(\omega_1+\omega_2\frac{z_{23}z_{24}}{z_{13}z_{14}}) \times\delta(\omega_3-\omega_2\frac{z_{12}z_{24}}{z_{13}z_{34}})\delta(\omega_4+\omega_2\frac{z_{12}z_{23}}{z_{14}z_{34}}). \tag{I.2}$$

We get some nice intuition for the above result when we consider setting $z_i = \bar{z}_i$ in the 3+1 four point Mellin paper, and removing the delta function for the real cross ratios since we are in one dimension less and on a locus where what would be the reality constraint of the cross ratios is already satisfied. Note that aside from the measure, the values of $\frac{\omega_i}{\omega_2}$ can be directly compared to 4.2 in the Mellin paper and agree on the $z_i = \bar{z}_i$ locus. Note also that the $12 \leftrightarrow 34$ channel corresponds to $x = \frac{z_{12}z_{34}}{z_{13}z_{24}} > 1$ as can be seen by demanding the localized $\omega_i$ be positive and noting that for instance $\mathrm{sgn}(\frac{\omega_3}{\omega_2}) = \mathrm{sgn}(\frac{z_{12}z_{24}}{z_{13}z_{34}}) = \mathrm{sgn}(\frac{z_{12}z_{34}}{z_{13}z_{24}}) = \mathrm{sgn}(x)$.

For a theory that is not scale invariant, one expects that the Mellin transform of a 4pt function will localize 3 of the frequency integrals. Having a massive intermediate propagator in a three point theory then gives hope for the final integral to be finite.

If I am considering charged massless scalars, then a $g\phi\varphi^*\varphi$ interaction can be set up to have only $s$ and $t$ channels. The desired amplitude is proportional to

$$\mathcal{A} \equiv g^2\left[\frac{1}{(p_1+p_2)^2+M^2}+\frac{1}{(p_1-p_3)^2+M^2}\right]\delta^{(3)}(p_1+p_2-p_3-p_4) \tag{I.3}$$

Then

$$\begin{aligned}\tilde{\mathcal{A}} &\equiv \prod_{i=1}^{4}\int d\omega_i\omega_i^{-\frac{1}{2}+i\lambda_i}\mathcal{A} \\ &= -\frac{2^{-2-i\sum\lambda_i}\pi}{M^{3-i\sum\lambda_i}}\mathrm{sech}(\frac{\pi}{2}\sum\lambda_i)\frac{g^2}{|z_{14}z_{13}z_{34}|}\theta\left(\frac{z_{12}z_{34}}{z_{13}z_{24}}-1\right) \\ &\times\left(\frac{z_{12}z_{23}z_{24}}{z_{14}}\right)^{\frac{1}{2}-\frac{i}{2}\sum\lambda_i}\left[\left(\frac{z_{12}}{z_{13}}\right)^{\frac{1}{2}-\frac{i}{2}\sum\lambda_i}+\left(-\frac{z_{24}}{z_{34}}\right)^{\frac{1}{2}-\frac{i}{2}\sum\lambda_i}\right] \\ &\times\left(-\frac{z_{23}z_{24}}{z_{13}z_{14}}\right)^{-\frac{1}{2}+i\lambda_1}\left(\frac{z_{12}z_{24}}{z_{13}z_{34}}\right)^{-\frac{1}{2}+i\lambda_3}\left(-\frac{z_{12}z_{23}}{z_{14}z_{34}}\right)^{-\frac{1}{2}+i\lambda_4}\end{aligned} \tag{I.4}$$

The support of the delta function is suppressed in the integrations and guarantees the localized values of $\omega_i$ are positive. Now the point is that this answer has non-singular support for the $z_i$ so that the 1 dimensional analog of Mack et al's decomposition theorem would say that we can write this in terms of conformal blocks. Meanwhile any statement of unitarity in 2+1 is guaranteed by the change of basis being consistent and starting with a momentum space $\mathcal{S}$-matrix that is already unitary. One should look at the conformal block decomposition for statements from the dual perspective.

## II. CONFORMAL BLOCK DECOMPOSITION

We will now use some equations from Jackiw & Pi 1205.0443. First their (2.4) gives the general form of a four point function

$$\langle\mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\mathcal{O}_3(z_3)\mathcal{O}_4(z_4)\rangle = \frac{F(x)}{z_{12}^{\Delta_1+\Delta_2}z_{34}^{\Delta_3+\Delta_4}z_{13}^{\Delta_{34}}z_{14}^{\Delta_{12}-\Delta_{34}}z_{24}^{-\Delta_{12}}} \tag{II.1}$$

where $\Delta_{ij} = \Delta_i - \Delta_j$, $x = \frac{z_{12}z_{34}}{z_{13}z_{24}}$ and $F(x)$ is a sum over blocks (2.5)

$$F(x) = \sum_i b_i B_i(x) \tag{II.2}$$

where (2.12)

$$B = x^{\Delta}\ {}_2F_1(\Delta-\Delta_{12}, \Delta+\Delta_{34}; 2\Delta; x) \tag{II.3}$$

For our scalars $\Delta_i = \frac{1}{2}+i\lambda_i$. We thus want to rewrite our (II.4) in terms of a sum/integral over $\Delta_i$. Factoring out the covariant piece a la (II.1), we find

$$\begin{aligned}F(x) =& -\frac{2^{-2-i\sum\lambda_i}\pi}{M^{3-i\sum\lambda_i}}\mathrm{sech}(\frac{\pi}{2}\sum\lambda_i)g^2(-1)^{-1+i\lambda_1+i\lambda_4}\,\mathrm{sgn}(z_{14}z_{13}z_{34}) \\ &\theta(x-1)x\left(\frac{1}{1-x}\right)^{\frac{1}{2}+\frac{i}{2}(-\lambda_1+\lambda_2+\lambda_3-\lambda_4)}\left[1+\left(-\frac{1}{x}\right)^{\frac{1}{2}-\frac{i}{2}\sum\lambda_i}\right].\end{aligned} \tag{II.4}$$

Keeping in mind the crossing vs. range of $x$ and avoiding the phase and $z_i$ dependent sign for the moment, we will now construct the conformal block decomposition of the simpler function:

$$\begin{aligned}f(x) &= x\left(\frac{1}{1-x}\right)^{\frac{1}{2}+\frac{i}{2}(-\lambda_1+\lambda_2+\lambda_3-\lambda_4)}\left[1+\left(-\frac{1}{x}\right)^{\frac{1}{2}-\frac{i}{2}\sum\lambda_i}\right] \\ &= x\left(\frac{1}{1-x}\right)^{\frac{1}{2}(1-\Delta_{12}+\Delta_{34})}\left[1+\left(-\frac{1}{x}\right)^{\frac{1}{2}-\frac{i}{2}\sum\lambda_i}\right].\end{aligned} \tag{II.5}$$

What then remains at this point is to find a principal series (or otherwise) decomposition of functions of the form $x^a(1-x)^{-\frac{1}{2}(1-\Delta_{12}+\Delta_{34})}$ a la (II.2), much like the author and S.H. Shao had attempted in the higher dimensional case, but now with the first hurdle of removing any singularity via shadows taken care of.