The Schrödinger Equation and Phase Space

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I arrive at the Schrödinger Equation given a particular form for probabilities in phase space.

In this paper, I explore what happens when the probability distribution for particles in phase space arises from:

$$\varphi(x,p,t) = \frac{1}{2\sqrt{2\pi\hbar}}[\Psi^*(x,t)\tilde{\Psi}(p,t)e^{\frac{ipx}{\hbar}} + c.c.] \tag{1}$$

for some complex $\Psi(x,t)$, such that expectation values of classical observables are found by integrating over this function: $\langle f(x,p)\rangle = \iint f(x,p)\varphi(x,p,t)\mathrm{d}x\mathrm{d}p$. Here "$+\ c.c.$" means that the complex conjugate is added. Equation 1 forces φ to be real, but allows it to be negative. It is completely specified by $\Psi(x,t)$, since $\tilde{\Psi}(p,t)$ is defined as the Fourier transform of $\Psi(x,t)$:

$$\tilde{\Psi}(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty}\Psi(y,t)e^{\frac{-ipy}{\hbar}}\mathrm{d}y. \tag{2}$$

To get some intuition for how φ relates to a probability distribution, notice that:

$$\begin{aligned}\mathbb{P}(x,t) &= \textstyle\int_{-\infty}^{+\infty}\varphi(x,p,t)\mathrm{d}p = |\Psi(x,t)|^2\\ \mathbb{P}(p,t) &= \textstyle\int_{-\infty}^{+\infty}\varphi(x,p,t)\mathrm{d}x = |\tilde{\Psi}(p,t)|^2\end{aligned} \tag{3}$$

would be the same definitions for the probability distributions in x and p if $\Psi(x,t)$ were taken to be the wave function from quantum mechanics. The integral over all x and p is defined to be normalized for all time, and φ approaches zero as $x,p\to\pm\infty$.

My "Hamilton's Equations of Motion" paper postulated that $\dot{\rho}=0$ in phase space for the Hamiltonian:

$$H(x,p) = \frac{p^2}{2m} + V(x). \tag{4}$$

This yields:

$$\begin{aligned}0 &= \partial_t\rho + \dot{x}\partial_x\rho + \dot{p}\partial_p\rho\\ &= \partial_t\rho + \tfrac{p}{m}\partial_x\rho - V'(x)\partial_p\rho.\end{aligned} \tag{5}$$

While Equation 1 gives:

$$\begin{aligned}\dot{\varphi} &= \tfrac{1}{2\sqrt{2\pi\hbar}}e^{\frac{ipx}{\hbar}}[\tfrac{\partial\Psi^*}{\partial t}\tilde{\Psi} + \Psi^*\tfrac{\partial\tilde{\Psi}}{\partial t} + \tfrac{p}{m}(\tfrac{\partial\Psi^*}{\partial x}\tilde{\Psi} + \tfrac{ip}{\hbar}\Psi^*\tilde{\Psi})\\ &\quad -V'(x)(\Psi^*\tfrac{\partial\tilde{\Psi}}{\partial p} + \tfrac{ix}{\hbar}\Psi^*\tilde{\Psi})] + c.c.\end{aligned} \tag{6}$$

The following calculations explore the consequence of restricting $\int\dot{\varphi}\mathrm{d}p = \int\dot{\rho}\mathrm{d}p = 0$. Using integration by parts:

$$\begin{aligned}p\tilde{\Psi}(p,t) &= \tfrac{1}{\sqrt{2\pi\hbar}}\textstyle\int_{-\infty}^{+\infty}p\Psi(y,t)e^{\frac{-ipy}{\hbar}}\mathrm{d}y\\ &= \tfrac{1}{\sqrt{2\pi\hbar}}\textstyle\int_{-\infty}^{+\infty}\Psi(y,t)\tfrac{-\hbar}{i}\partial_y[e^{\frac{-ipy}{\hbar}}]dy\\ &= \tfrac{1}{\sqrt{2\pi\hbar}}\textstyle\int_{-\infty}^{+\infty}\tfrac{\hbar}{i}\partial_y[\Psi(y,t)]e^{\frac{-ipy}{\hbar}}\mathrm{d}y\end{aligned} \tag{7}$$

since the boundary term is zero. Similarly,

$$\partial_p\tilde{\Psi}(p,t) = \tfrac{1}{\sqrt{2\pi\hbar}}\textstyle\int_{-\infty}^{+\infty}[\tfrac{-iy}{\hbar}]\Psi(y,t)e^{\frac{-ipy}{\hbar}}\mathrm{d}y. \tag{8}$$

These relations cause the $\dot{p}\partial_p\varphi$ term to vanish when integrated over p. The first two terms in Equation 5 yield:

$$\partial_t[\Psi^*\Psi] = -\partial_x[\frac{\hbar}{2mi}(\Psi^*\partial_x\Psi - \Psi\partial_x\Psi^*)]. \tag{9}$$

If $\mathbb{P}(x,t)=\mathbb{P}(x)$, so that $\Psi(x,t)=\psi(x)e^{if(t)}$ for some time-dependent phase, then $\psi^*\partial_x\psi - \psi\partial_x\psi^*$ must be a constant. Using the limit that $\psi\to 0$ for $x\to\pm\infty$ means this constant is zero: $\psi^*\partial_x\psi$ is real. An arbitrary $\psi(x)$ can be written as $\psi = A(x)e^{ig(x)}$ for some real functions $A(x)$ and $g(x)$:

$$\begin{aligned}0 &= \mathrm{Im}[\psi^*\partial_x\psi]\\ &= \mathrm{Im}[A(x)(A'(x) + iA(x)g'(x))]\end{aligned} \tag{10}$$

so $g'(x)=0$ and $\psi(x)$ is real up to a constant phase.

Next, consider taking the expectation value of $H(x,p)$ as a function of x by integrating over p:

$$\int_{-\infty}^{+\infty}H(x,p)\varphi\mathrm{d}p = \frac{1}{2}[\Psi^*(\frac{-\hbar^2}{2m}\partial_x^2 + V(x))\Psi + c.c.] \tag{11}$$

which for the time-independent case becomes:

$$\int_{-\infty}^{+\infty}H(x,p)\varphi\mathrm{d}p = \psi(\frac{-\hbar^2}{2m}\partial_x^2 + V(x))\psi. \tag{12}$$

There will be some ψ_n for which:

$$\int_{-\infty}^{+\infty}H(x,p)\varphi_n\mathrm{d}p = E_n\mathbb{P}_n(x). \tag{13}$$

From Equation 13, these eigenfunctions ψ_n satisfy the differential equation:

$$\frac{-\hbar^2}{2m}\partial_x^2\psi_n + V(x)\psi_n = E_n\psi_n \tag{14}$$

away from $\psi_n=0$, and continue to satisfy Equation 14 if restricted to having $\partial_x^2\psi_n=0$ when $\psi_n=0$.

Letting $\Psi_n(x,t)=\psi_n(x)e^{if_n(t)}$, plug $\Psi=\Psi_1+\Psi_2$ into Equation 9:

$$\begin{aligned}\tfrac{1}{i\hbar}[\Psi^*\tfrac{-\hbar^2}{2m}\partial_x^2\Psi] + c.c. &= -2\psi_1\psi_2[f_1' - f_2']\sin(f_1-f_2)\\ &= \tfrac{2}{\hbar}\psi_1\psi_2[E_1-E_2]\sin(f_1-f_2)\end{aligned} \tag{15}$$

If a time translation of Ψ_n is still a solution, let $\Psi_2(x,t) = \psi_1(x)e^{if_1(t+\Delta t)}$. Since $E_2=E_1$, and $f_1(t)-f_1(t+\Delta t) = n\pi$ for $n\in\mathbb{Z}$ would not hold for all Δt unless f_1 is constant, we must have $f_1'(t)=f_1'(t+\Delta t)$: the phase is linear in time. Equation 15 is consistent with $f_n(t) = -E_nt/\hbar$ up to a constant phase. This gives:

$$E_n\Psi_n = i\hbar\partial_t\Psi_n \tag{16}$$

If we restrict Ψ to linear combinations of Ψ_n, we see that the Schrödinger Equation is obeyed:

$$i\hbar\partial_t\Psi = \frac{-\hbar^2}{2m}\partial_x^2\Psi + V(x)\Psi. \tag{17}$$