Hamilton's Equations of Motion

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I motivate Hamilton's equations of motion using a geometrical picture of contours in phase space. The following considers a single cartesian coordinate x with conjugate momentum p.

I. POSTULATES

1. There exists a function $H(x,p)$ which is constant along a particle's trajectory in phase space and is time-independent.

2. The momentum p is defined as $p = m\dot{x}$.

3. Motion within phase space is characterized by incompressible fluid flow, so that the phase space velocity is divergence-less: $\vec{\nabla}\cdot\vec{v} = 0$.

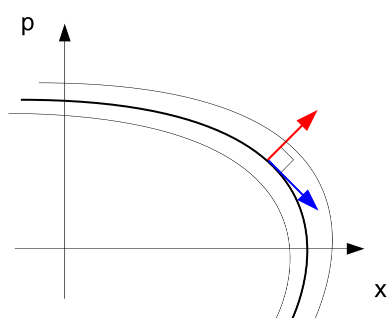

FIG. 1. Illustration of contours of $H(x,p)$ in phase space.

II. DERIVATION

At any position on a contour of $H(x,p)$, the gradient:

$$\vec{\nabla}H = \frac{\partial H}{\partial x}\hat{x} + \frac{\partial H}{\partial p}\hat{p} \tag{1}$$

points perpendicular to this contour. This is represented by the red arrow in Figure 1. I can define a vector $\vec{\eta}$ perpendicular to $\vec{\nabla}H$ in the (x,p)-plane:

$$\vec{\eta} = \frac{\partial H}{\partial p}\hat{x} - \frac{\partial H}{\partial x}\hat{p} \tag{2}$$

represented by the blue arrow in Figure 1. Being perpendicular to the gradient, which is perpendicular to the contour, we find that $\vec{\eta}$ points along the contour of $H(x,p)$.

Since a particle's motion is restricted to contours of H by Postulate 1, its instantaneous velocity in phase space will be parallel to the contour it is on and, thus, $\vec{\eta}$. The magnitude of the velocity is not fixed; however, it can be handled by multiplying $\vec{\eta}$ in Equation 2 by an unknown function $\alpha(x,p)$, so that:

$$\vec{v} = \dot{x}\hat{x} + \dot{p}\hat{p} = \alpha(x,p)\left[\frac{\partial H}{\partial p}\hat{x} - \frac{\partial H}{\partial x}\hat{p}\right]. \tag{3}$$

From Postulate 2, $\dot{x} = \frac{p}{m}$, so that we could eliminate $\alpha(x,p)$:

$$\vec{v} = \dot{x}\hat{x} + \dot{p}\hat{p} = \frac{p}{m}\left[\hat{x} - \frac{\frac{\partial H}{\partial x}}{\frac{\partial H}{\partial p}}\hat{p}\right] \tag{4}$$

when $\frac{\partial H}{\partial p} \neq 0$. In what follows, I use the form of $\vec{v}$ in Equation 3 and Postulate 2 to verify that α is a function of x and p that does not depend explicitly on time, since $\dot{x} = \frac{p}{m}$ sets the overall speed.

Using Postulate 3, the divergence of the phase space velocity field is zero, giving:

$$\begin{aligned}\vec{\nabla}\cdot\vec{v} &= \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{p}}{\partial p} \\ &= \frac{\partial \alpha}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial \alpha}{\partial p}\frac{\partial H}{\partial x} = 0\end{aligned} \tag{5}$$

This expression, which is equivalent to saying $\{\alpha, H\} = 0$, tells us that α is a constant of the motion using geometrical logic. It is equivalent to the statement that $\vec{\nabla}\alpha$ and $\vec{\nabla}H$ are parallel if both are nonzero since:

$$\begin{aligned}\vec{\nabla}\alpha \times \vec{\nabla}H &= \begin{vmatrix} \hat{x} & \hat{p} & \hat{\xi} \\ \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial p} & 0 \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial p} & 0 \end{vmatrix} \\ &= \left[\frac{\partial \alpha}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial \alpha}{\partial p}\frac{\partial H}{\partial x}\right]\hat{\xi} = \vec{0}\end{aligned} \tag{6}$$

where a third dimension ξ has been added for convenience which is perpendicular to the (x,p)-plane. If the gradients of $\alpha(x,p)$ and $H(x,p)$ are everywhere parallel, then the contours of $\alpha(x,p)$ and $H(x,p)$ will coincide since the contour of a function is at each point perpendicular to its gradient. A contour of H is thus also a contour of α. Since $\alpha(x,p)$ is constant along a particle's path, setting $\alpha = 1$ amounts to rescaling the value of $H(x,p)$ on each contour, which does not change the implications of Postulate 1. Equation 3 thus gives us Hamilton's equations of motion:

$$\begin{aligned}\dot{x} &= +\frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial x}\end{aligned} \tag{7}$$

from which Postulate 2 gives us:

$$H(x,p) = \frac{p^2}{2m} + V(x) \tag{8}$$

for some function $V(x)$ interpreted as the potential.