Gaussian Measures and the QM Oscillator

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In this paper, I show how probability densities associated with a Gaussian field can be expressed in terms of the Boltzmann heat kernel. The $N \leq 2$ calculations are based off of the work of Arthur Jaffe, while the proof of his postulate for general N is original.

In "Fields with a Gaussian Measure," I found:

$$\begin{aligned} \rho_t(x) &\equiv \int \prod_{i=1}^{N} \delta(\Phi(t_i) - x_i) \mathrm{d}\mu_{\mathrm{c}} \\ &= \frac{1}{\sqrt{(2\pi)^n \det \mathrm{C}}} e^{-\frac{1}{2}\mathrm{x}^\top \mathrm{C}^{-1}\mathrm{x}} \end{aligned} \tag{1}$$

where $\mathrm{C}_{ij} = C(t_i - t_j) = \frac{1}{2m} e^{-m|t_i - t_j|}$. Now, consider the operator:

$$H_0 = \frac{1}{2}\left(-\frac{d^2}{dx^2} + m^2 x^2 - m\right) \tag{2}$$

which can be thought of as the Hamiltonian for the simple harmonic oscillator with position coordinate scaled to have unit mass, and frequency $\omega = m$. The spectrum is $m\mathbb{Z}^+$ and the ground state is given by:

$$\Omega_0(x) = \left(\frac{m}{\pi}\right)^{1/4} e^{-\frac{mx^2}{2}} \tag{3}$$

For $t > 0$, the Boltzmann integral kernel gives the evolution:

$$\left(e^{-tH_0} f\right)(x) = \int_{-\infty}^{\infty} \mathcal{B}_t(x, x') f(x') \mathrm{d}x' \tag{4}$$

For $N = 1$, $\rho_t(x) = \Omega_0(x)^2$. When $N > 1$, it is convenient to define $\mathbb{C} = 2mC$ so that $\mathbb{C}_{ij} = e^{-m|t_i - t_j|}$. Then:

$$\rho_t(x) = \left(\frac{m}{\pi}\right)^{N/2} \frac{e^{-m\mathrm{x}^\top \mathbb{C}^{-1}\mathrm{x}}}{\sqrt{\det \mathbb{C}}} \tag{5}$$

Here, I will consider $t_1 < ... < t_N$. For $N = 2$ explicitly inverting

$$\mathbb{C}_2 \equiv \begin{pmatrix} 1 & e^{-m(t_2 - t_1)} \\ e^{-m(t_2 - t_1)} & 1 \end{pmatrix} \tag{6}$$

gives an expression for ρ in terms of $\mathcal{B}$:

$$\rho_{t_1, t_2}(x_1, x_2) = \Omega_0(x_1) \mathcal{B}_{t_2 - t_1}(x_1, x_2) \Omega_0(x_2) \tag{7}$$

I can now find an expression for general N using induction. Writing $\mathbb{C}_N$ in blocks:

$$\mathbb{C}_N \equiv \begin{pmatrix} \mathbb{C}_{N-1} & v \\ v^\top & 1 \end{pmatrix} \tag{8}$$

where $v^\top = (e^{-m(t_N - t_1)} ... e^{-m(t_N - t_{N-1})})$, leads to an expression for the inverse:

$$\mathbb{C}_N^{-1} \equiv \begin{pmatrix} \mathbb{C}_{N-1}^{-1} + \frac{(\mathbb{C}_{N-1}^{-1} v)(\mathbb{C}_{N-1}^{-1} v)^\top}{\mu} & -\frac{(\mathbb{C}_{N-1}^{-1} v)}{\mu} \\ -\frac{(\mathbb{C}_{N-1}^{-1} v)^\top}{\mu} & \frac{1}{\mu} \end{pmatrix} \tag{9}$$

where $\mu = 1 - v^\top (\mathbb{C}_{N-1}^{-1} v)$. Rather than inverting $\mathbb{C}_{N-1}$, my expression for ρ_N in terms of ρ_{N-1} will only need the product $(\mathbb{C}_{N-1}^{-1} v)$. Because the inverse exists, it is equivalent to finding ξ such that $v = \mathbb{C}_{N-1}\xi$. Since the last column of $\mathbb{C}_{N-1}$ is $(e^{-m(t_{N-1} - t_1)} ... 1)$, I find that:

$$(\mathbb{C}_{N-1}^{-1} v)_j = e^{-m(t_N - t_{N-1})} \delta_{j, N-1} \tag{10}$$

which gives:

$$\begin{aligned} \mu &= 1 - e^{-2m(t_N - t_{N-1})} \\ \mathrm{x}_N^\top \mathbb{C}_N^{-1} \mathrm{x}_N &= \mathrm{x}_{N-1}^\top \mathbb{C}_{N-1}^{-1} \mathrm{x}_{N-1} + \frac{1}{\mu}[x_N^2 + e^{-2m(t_N - t_{N-1})} x_{N-1}^2 \\ &\quad - 2x_N x_{N-1} e^{-m(t_N - t_{N-1})}] \end{aligned} \tag{11}$$

In terms of ρ_{N-1}, one thus finds:

$$\begin{aligned} \rho_N &= \left(\frac{m}{\pi}\right)^{\frac{1}{2}} \left(\frac{m}{\pi}\right)^{\frac{N-1}{2}} \frac{e^{-m\mathrm{x}_{N-1}^\top \mathbb{C}_{N-1}^{-1} \mathrm{x}_{N-1}}}{\sqrt{\det \mathbb{C}_{N-1}}} \frac{e^{-m\Delta(\mathrm{x}^\top \mathbb{C}^{-1}\mathrm{x})}}{\sqrt{\mu}} \\ &= \rho_{N-1} \rho_{t_N, t_{N-1}}(x_N, x_{N-1}) \Omega_0(x_{N-1})^{-2} \\ &= \rho_{N-1} \Omega_0(x_{N-1})^{-1} \mathcal{B}_{t_N - t_{N-1}}(x_{N-1}, x_N) \Omega_0(x_N) \end{aligned} \tag{12}$$

The expressions for $N = 1$ and $N = 2$ are both consistent with the following expression for general N:

$$\begin{aligned} \rho_N = & \ \Omega_0(x_1) \mathcal{B}_{t_2 - t_1}(x_1, x_2) \mathcal{B}_{t_3 - t_2}(x_2, x_3) ... \\ & ... \mathcal{B}_{t_N - t_{N-1}}(x_{N-1}, x_N) \Omega_0(x_N) \end{aligned} \tag{13}$$

where $t_1 < ... < t_N$.