From Gaussian Measures to SDE

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(Dated: March 31, 2014)

In this paper, I show how a field with gaussian measure can give the Schwinger-Dyson Equations of QFT in Wick rotated time.

In my paper on "Fields with a Gaussian Measure," I found that:

$$\begin{aligned}&\int \prod_{i=1}^{n} \delta(\Phi(t_i) - x_i)\mathrm{d}\mu_\mathrm{c} \\ &= \frac{1}{\sqrt{(2\pi)^n \det \mathrm{C}}} e^{-\frac{1}{2}\mathrm{x}^\top \mathrm{C}^{-1}\mathrm{x}}\end{aligned} \tag{1}$$

where $\mathrm{C}_{ij} = C(t_i - t_j) = \frac{1}{2m}e^{-m|t_i - t_j|}$. The function $C(t_i - t_j)$ solves:

$$([-\nabla^2 + m^2]^{-1} f)(t) = \int C(t - t')f(t')dt' \tag{2}$$

Pluging in $f(t') = \delta(t')$ gives:

$$C(t) = ([-\nabla^2 + m^2]^{-1}\delta)(t) \tag{3}$$

So that the matrix elements can be expressed as: $\mathrm{C}_{ij} = C(t_i - t_j) = ([-\nabla^2 + m^2]^{-1}\delta)(t_i - t_j)$. This means that the entries are the value of the impulse response of the operator.

In a discrete time basis, the delta function becomes the identity operator. In this basis, $\mathrm{x}^\top \mathrm{C}^{-1}\mathrm{x}$ is transformed so that $x_i = x(t_i)$. Then:

$$\mathrm{x}^\top \mathrm{C}^{-1}\mathrm{x} = \mathrm{x}^\top(t_i)[-\nabla^2 + m^2]_{ij}\mathrm{x}(t_j) \tag{4}$$

In the continuum time limit $n \to \infty$, this becomes:

$$\mathrm{x}^\top \mathrm{C}^{-1}\mathrm{x} \propto \int \mathrm{d}t\{\mathrm{x}(t)[-\nabla^2 + m^2]\mathrm{x}(t)\} \tag{5}$$

So that the original expression becomes:

$$\begin{aligned}&\int \delta(\Phi(t) - x(t))\mathrm{d}\mu_\mathrm{c} \\ &\propto \sqrt{\det[-\nabla^2 + \mathrm{m}^2]}e^{-\frac{1}{2}\int \mathrm{d}t\{\mathrm{x}(t)[-\nabla^2 + m^2]\mathrm{x}(t)\}}\end{aligned} \tag{6}$$

This expression agrees with the conventional definition of the path integral normalization if we set the constant of proportionality to 1 and think of the delta function in Φ as a product over delta functions at each time:

$$\begin{aligned}1 &= \int \mathcal{D}x\sqrt{\det[-\nabla^2 + m^2]}e^{-\frac{1}{2}\int \mathrm{d}t\{\mathrm{x}(t)[-\nabla^2 + m^2]\mathrm{x}(t)\}} \\ &= \int \mathcal{D}x \int \delta(\Phi(t) - x(t))\mathrm{d}\mu_\mathrm{c} \\ &= \int \left\{\int \mathcal{D}x\delta(\Phi(t) - x(t))\right\}\mathrm{d}\mu_\mathrm{c} \\ &= \int \mathrm{d}\mu_\mathrm{c} \\ &= 1\end{aligned} \tag{7}$$

I can now write the Euclidean action for a free field $x(t)$ of mass m in terms of the Gaussian field $\Phi(t)$:

$$S = -\ln\left[\frac{\int \delta(\Phi(t) - x(t))\mathrm{d}\mu_\mathrm{c}}{\sqrt{\det[-\nabla^2 + m^2]}}\right] \tag{8}$$

This expression allows one to show how adding an interaction term $\int \mathrm{d}t\mathcal{L}_{int}[\mathrm{x}(t)]$ amounts to rescaling the gaussian measure:

$$\begin{aligned}S + \int \mathrm{d}t_0\mathcal{L}_{int}[\mathrm{x}(t_0)] &= -\ln\left[e^{-\int \mathrm{d}t_0\mathcal{L}_{int}[\mathrm{x}(t_0)]}\frac{\int \delta(\Phi(t) - x(t))\mathrm{d}\mu_\mathrm{c}}{\sqrt{\det[-\nabla^2 + m^2]}}\right] \\ &= -\ln\left[\frac{\int \delta(\Phi(t) - x(t))e^{-\int \mathrm{d}t_0\mathcal{L}_{int}[\Phi(t_0)]}\mathrm{d}\mu_\mathrm{c}}{\sqrt{\det[-\nabla^2 + m^2]}}\right]\end{aligned} \tag{9}$$

Here, $x(t)$ appears as a field from QFT. I will show that the expectation value of $x(t)x(t')$ evaluated in terms of the gaussian measure field Φ obeys the Schwinger-Dyson equations of QFT for the time ordered product: $\langle 0|T\{x(t)x(t')\}|0\rangle$.

$$\begin{aligned}\langle x(t)x(t')\rangle &= \frac{\int \mathcal{D}x\{x(t)x(t')\}e^{-\frac{1}{2}\int \mathrm{d}t\{\mathrm{x}(t)[-\nabla^2 + m^2]\mathrm{x}(t) - \mathcal{L}_{int}\}}}{\int \mathcal{D}x e^{-\frac{1}{2}\int \mathrm{d}t\{\mathrm{x}(t)[-\nabla^2 + m^2]\mathrm{x}(t) - \mathcal{L}_{int}\}}} \\ &= \frac{\int \Phi(t)\Phi(t')e^{-\int \mathrm{d}t_0\mathcal{L}_{int}[\Phi(t_0)]}\mathrm{d}\mu_\mathrm{c}}{\int e^{-\int \mathrm{d}t_0\mathcal{L}_{int}[\Phi(t_0)]}\mathrm{d}\mu_\mathrm{c}}\end{aligned} \tag{10}$$

Because the Φ fields have a gaussian measure, the integral will factor into a product over all possible permutations of two point functions. The exponential of the interaction term can be written as a sum over products of $\mathcal{L}_{int}$:

$$e^{-\int \mathrm{d}t_0\mathcal{L}_{int}} = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\int \mathrm{d}t_1...\mathrm{d}t_m\mathcal{L}_{int}[\Phi(t_1)]...\mathcal{L}_{int}[\Phi(t_m)] \tag{11}$$

The sum over all possible two point correlation functions can be factored into a sum over the product of $\Phi(t)$ with one other field, times the expectation value of the remaining fields. As a first step in this decomposition:

$$\begin{aligned}&\langle \Phi(t)\Phi(t')e^{-\int \mathrm{d}t_0\mathcal{L}_{int}[\Phi(t_0)]}\rangle_c \\ &\quad = \langle \Phi(t)\Phi(t')\rangle_c\langle e^{-\int \mathrm{d}t_0\mathcal{L}_{int}[\Phi(t_0)]}\rangle_c + ...\end{aligned} \tag{12}$$

In the remaining terms, $\Phi(t)$ is contracted with a $\Phi(t_i)$ from the expansion of the $\mathcal{L}_{int}$ term in Equation 11. When acted on with $(-\nabla^2 + m^2)_t$:

$$(-\nabla^2 + m^2)_t\langle \Phi(t)\Phi(t_i)\rangle_c = \delta(t - t_i) \tag{13}$$

Since the t_i is integrated over, this amounts to removing one factor of Φ from each term in the $\mathcal{L}_{int}$ expansion and replacing its argument with t. This is equivalent to having a factor of:

$$\frac{\delta}{\delta\Phi(t)}e^{-\int \mathrm{d}t_0\mathcal{L}_{int}[\Phi(t_0)]} = -\frac{\delta\mathcal{L}_{int}}{\delta\Phi(t)}e^{-\int \mathrm{d}t_0\mathcal{L}_{int}[\Phi(t_0)]} \tag{14}$$

in the remaining term multiplying $\Phi(t')$. As a result:

$$\begin{aligned}(-\nabla^2 + m^2)_t\langle x(t)x(t')\rangle &= (-\nabla^2 + m^2)_t\langle \Phi(t)\Phi(t')\rangle_c \\ &\quad - \frac{\langle \frac{\delta\mathcal{L}_{int}}{\delta\Phi(t)}\Phi(t')e^{-\int \mathrm{d}t_0\mathcal{L}_{int}[\Phi(t_0)]}\rangle_c}{\langle e^{-\int \mathrm{d}t_0\mathcal{L}_{int}[\Phi(t_0)]}\rangle_c} \\ &= \delta(t - t') - \langle \frac{\delta\mathcal{L}_{int}}{\delta x(t)}x(t')\rangle\end{aligned} \tag{15}$$

This is the Schwinger-Dyson Equation in Wick rotated time. Note if $\mathcal{L}$ is replaced by $\mathcal{L}/\hbar$, and $(-\nabla^2+m^2)C(t-t') \to \hbar\delta(t-t')$, the contact term associated with "quantum corrections" will vanish for $\hbar \to 0$, while the $\frac{\delta\mathcal{L}}{\delta\Phi}$ term will get multiplied by $\hbar/\hbar = 1$ and will remain, giving the "classical result." In these calculations, the difference between classical and quantum field theory results for this one-dimensional example were shown to arise from having a field that is a gaussian random variable in time (with width set by $\hbar$), rather than being differentiable. In this context, the significance of $\hbar$ is as the limiting variance of a free field in the vacuum (when interaction terms become irrelevant). The independence of this variance on the mass is interesting, but can often be absorbed into the normalization of the fields. The gaussian nature of the time correlations would seem to relate to the ubiquity of gaussians following the central limit theorem.