Soft Theorems and Symmetry

Sabrina Gonzalez Pasterski
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I examine the connection between symmetries and soft factors for the case of graviton scattering.

In QFT, soft theorems describe the effect of adding an additional low momentum particle to an existing process and observing the change in the S matrix as this new particle's momentum is taken to zero. In the particular case where this particle is added to an external line, the change in the S matrix amounts to adding an interaction vertex factor and an extra factor of the propagator for the line to which the soft particle is attached.

This can be more clearly seen by considering the momentum space correlation function for the interaction. As the external particles for a given process go on shell:

$$G(p_1, ...p_n) \propto \left[\prod_{j=1}^{n} \frac{-i}{p_j^2 + m_j^2 - i\epsilon}\right] S(p_1, ...p_n) \tag{1}$$

so that there are poles corresponding to each incoming and outgoing particle. When a massless soft particle of momentum q is also emitted, the momentum space correlation function becomes:

$$G(q, p_1, ...p_n) \propto \frac{-i}{q^2 - i\epsilon}\left[\prod_{j=1}^{n} \frac{-i}{p_j^2 + m_j^2 - i\epsilon}\right] \tilde{S}(q, p_1, ...p_n) \tag{2}$$

The goal is to relate $S(p_1, ...p_n)$ to $\tilde{S}(q, p_1, ...p_n)$.

Consider the case where a massless ψ particle of momentum q is attached to an outgoing ϕ particle with momentum p_1 and mass m_1, a Feynman diagram approach shows that the difference between the two matrix elements S and $\tilde{S}$ is and overall factor of the propagator of a ϕ particle with momentum $p_1 + q$ and the vertex factor associated with the $\psi\phi\phi$ interaction, which I will denote $\mathcal{V}(\psi J_\phi)$:

$$\begin{aligned} \tilde{S}(q, p_1, ...p_n) &\approx \tfrac{-i}{(p_1+q)^2+m_1^2-i\epsilon}\mathcal{V}(\psi J_\phi) S(p_1, ...p_n) \\ &= \tfrac{-i}{2p_1\cdot q}\mathcal{V}(\psi J_\phi) S(p_1, ...p_n) \end{aligned} \tag{3}$$

Two important considerations allowed me to write the new matrix element in this form: 1. Since q is small, I assumed that changing p_1 to p_1+q did not affect the rest of the diagram. (Sometimes, derivations will change the momentum of the new external leg instead. Either case requires the approximation that the ϕ particle is nearly on shell both before and after the ψ emission.) 2. the fact that q is massless and on-shell resulted in the dot-product form of the denominator, since the p_1^2 cancelled the m_1^2.

What remains is to calculate the vertex factor due to the ψ particle interacting with the J_ϕ current. As $q \to 0$, only terms of up to $\mathcal{O}(q)$ in $\mathcal{V}(\psi J_\phi)$ will survive. This amounts to considering interaction terms in the Lagrangian that have only 0 or 1 derivatives of ψ.

Gauge invariance restricts the form of the interaction terms allowed. Consider electromagnetism as an example. If the photon field A_μ couples to matter via $A_\mu J^\mu$, then sending $A_\mu \to \nabla_\mu \lambda$ gives:

$$\begin{aligned} A_\mu J^\mu &\to (\nabla_\mu \lambda) J^\mu \\ &\to -\lambda \nabla_\mu J^\mu \end{aligned} \tag{4}$$

where the second line is after integrating by parts within the Lagrangian. Since this must be zero for any function λ, one concludes $\nabla_\mu J^\mu = 0$.

For the case where ψ represents a soft graviton, similar gauge invariance allows us to predict the form of the interaction vertices in the Lagrangian. Here the dynamical ψ field is $h_{\mu\nu}$ where $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$. Interaction terms in the Lagrangian involving $h_{\mu\nu}$ must be completely contracted to preserve Lorentz invariance.

First, consider a term with no derivatives of $h_{\mu\nu}$: $\mathcal{L}^0_{int} = h_{\mu\nu} S^{\mu\nu}$. Gauge invariance requires that sending $h_{\mu\nu} \to \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$ gives zero:

$$\begin{aligned} 0 &= (\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) S^{\mu\nu} \\ &= -\xi_\nu (\nabla_\mu S^{\mu\nu}) - \xi_\mu (\nabla_\nu S^{\mu\nu}) \\ &= -2\xi_\nu \nabla_\mu S^{(\mu\nu)} \end{aligned} \tag{5}$$

from this one concludes $\nabla_\mu S^{(\mu\nu)} = 0$. Since only the symmetric part of $S_{\mu\nu}$ remains after contracting with $h_{\mu\nu}$, we find that $h_{\mu\nu}$ couples to a conserved rank 2 tensor. A natural candidate is $\mathcal{L}^0_{int} \propto h_{\mu\nu} T_M^{\mu\nu}$. Where $T_M^{\mu\nu}$ is the matter stress-energy tensor.

Now consider terms with a single derivative of $h_{\mu\nu}$. For a transversely polarized graviton field a $\partial_\nu h^{\mu\nu}$ term within $\mathcal{L}^1_{int}$ would result in a $q_\nu \epsilon^{\mu\nu} = 0$ within the vertex factor associated to this interaction. We can thus hypothesize an interaction term of the form: $\mathcal{L}^1_{int} = \partial_\lambda h_{\mu\nu} S^{\mu\nu\lambda}$. Gauge invariance implies:

$$\begin{aligned} 0 &= \partial_\lambda (\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) S^{\mu\nu\lambda} \\ &= -(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) \partial_\lambda S^{\mu\nu\lambda} \\ &\approx \xi_\nu (\partial_\mu \partial_\lambda S^{\mu\nu\lambda}) + \xi_\mu (\partial_\nu \partial_\lambda S^{\mu\nu\lambda}) \\ &= 2\xi_\nu \partial_\mu \partial_\lambda S^{(\mu\nu)\lambda} \end{aligned} \tag{6}$$

where I have replaced covariant derivatives with partial derivatives in my weak gravity approximation to avoid ambiguities in the product rule for the covariant derivative acting on a non-tensor object. The conserved angular momentum tensor has the desired structure: $\partial_\mu M^{\mu\nu\lambda} = 0$ where $M^{\mu\nu\lambda} = x^\nu T_M^{\mu\lambda} - x^\lambda T_M^{\mu\nu}$. Note also that the antisymmetry of this tensor in $\nu \to \lambda$ would give:

$$\begin{aligned}\nabla_\lambda \nabla_\mu S^{\mu\nu\lambda} &= \partial_\lambda \nabla_\mu S^{\mu\nu\lambda} + \Gamma^\nu_{\lambda\sigma}\nabla_\mu S^{\mu\sigma\lambda} + \Gamma^\lambda_{\lambda\sigma}\nabla_\mu S^{\mu\nu\sigma} \\ &= \partial_\lambda \nabla_\mu S^{\mu\nu\lambda}\end{aligned} \tag{7}$$

at linear order in $h_{\mu\nu}$ if I choose a traceless gauge, since the second term is zero by the antisymmetry of S, while $\Gamma^\lambda_{\lambda\sigma} = \frac{1}{2}g^{\rho\tau}\partial_\sigma g_{\rho\tau} = \kappa\partial_\sigma h^\rho_\rho + \mathcal{O}(h^2)$ which is zero if $h^\rho_\rho = 0$. From Equation 6, it is thus natural to hypothesize in interaction term: $\mathcal{L}^1_{int} \propto \partial_\lambda h_{\mu\nu} M^{\mu\nu\lambda}$.

I will now show that these Lagrangian interaction terms give expected vertex factors for a massless scalar field. The stress energy tensor for a massless scalar field is:

$$T^{\mu\nu}_M \propto (g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} - g^{\mu\nu}g^{\alpha\beta})\partial_\alpha\phi\partial_\beta\phi \tag{8}$$

so that within the contractions for $\mathcal{L}^0_{int}$ and $\mathcal{L}^1_{int}$, I can take $T^{\mu\nu}_M \propto \partial^\mu\phi\partial^\nu\phi$. This gives:

$$\mathcal{L}^0_{int} \propto h_{\mu\nu}\partial^\mu\phi\partial^\nu\phi \tag{9}$$

and

$$\begin{aligned}\mathcal{L}^1_{int} &\propto \partial_\lambda h_{\mu\nu}(x^\nu\partial^\mu\phi\partial^\lambda\phi - x^\lambda\partial^\mu\phi\partial^\nu\phi) \\ &= \partial_\lambda h_{\mu\nu}(x^\nu\partial^\lambda\phi - x^\lambda\partial^\nu\phi)\partial^\mu\phi\end{aligned} \tag{10}$$

Partial derivatives will pull down factors of the corresponding field's momentum, while $h_{\mu\nu}$ will go to the the polarization vector $\epsilon_{\mu\nu}$ in the vertex factor:

$$\begin{aligned}\mathcal{V}^0(hJ_\phi) &\propto \epsilon_{\mu\nu}p^\mu p^\nu \\ \mathcal{V}^1(hJ_\phi) &\propto q_\lambda\epsilon_{\mu\nu}S^{\nu\lambda}p^\mu\end{aligned} \tag{11}$$

So that the soft factors corresponding to the addition of a soft graviton to an external line are:

$$\boxed{\begin{aligned}X^0_q &\propto \frac{\epsilon_{\mu\nu}p^\mu p^\nu}{p\cdot q} \\ X^1_q &\propto \frac{q_\lambda\epsilon_{\mu\nu}S^{\nu\lambda}p^\mu}{p\cdot q}\end{aligned}} \tag{12}$$