Fields with a Gaussian Measure

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(Dated: March 17, 2014)

In this paper, I calculate $\int \prod_{i=1}^n \delta(\Phi(t_i) - x_i)\mathrm{d}\mu_\mathrm{c}$. This corresponds to specifying the value of a field Φ defined by a gaussian measure $\mathrm{d}\mu_\mathrm{c}$ to take the values x_i at a set of n distinct times t_i.

In order to calculate $\int \prod_{i=1}^n \delta(\Phi(t_i) - x_i)\mathrm{d}\mu_\mathrm{c}$, I begin by calculating:

$$\int \mathrm{d}\mu_\mathrm{c} e^{i\int \mathrm{d}v \Phi(v) g(v)} = \int \mathrm{d}\mu_\mathrm{c} \left\{ \sum_{n=0}^{\infty} \frac{i^n}{n!} \left(\int \mathrm{d}v \Phi(v) g(v) \right)^n \right\}. \tag{1}$$

A generic term in the expansion of the exponential can be rewritten in terms multiple integration variables v_i:

$$\int \mathrm{d}\mu_\mathrm{c} \left(\int \mathrm{d}v \Phi(v) g(v) \right)^n = \int \mathrm{d}\mu_\mathrm{c} \prod_{i=1}^{n} \mathrm{d}v_i g(v_i) \Phi(v_i). \tag{2}$$

The integral over $\mathrm{d}\mu_\mathrm{c}$ will be non-zero only when n is even, in which case it takes the form:

$$\int \mathrm{d}\mu_\mathrm{c} \prod_{i=1}^{n} \Phi(v_i) = \sum_{pairings} C(v_{i_1} - v_{i_2}) ... C(v_{i_{n-1}} - v_{i_n}) \tag{3}$$

where $C(v_i - v_j) = \frac{1}{2m} e^{-m|v_i - v_j|}$ and there are $(n-1)!!$ possible pairings. Since each term in the sum involves a product of $\frac{n}{2}$ two-point correlation functions and the $g(v_i)$ are symmetric under exchange of v_i:

$$\begin{aligned} &\int \mathrm{d}\mu_\mathrm{c} \prod_{i=1}^n \mathrm{d}v_i g(v_i) \Phi(v_i) \\ &\quad = (n-1)!! \left(\int \mathrm{d}v_1 \mathrm{d}v_2 g(v_1) g(v_2) C(v_1 - v_2) \right)^{\frac{n}{2}}. \end{aligned} \tag{4}$$

The original expression then simplifies to:

$$\begin{aligned} &\int \mathrm{d}\mu_\mathrm{c} e^{i\int \mathrm{d}v \Phi(v) g(v)} \\ &\quad = \sum_{n=0}^{\infty} \frac{i^{2n}}{2n!} (2n-1)!! \left(\int \mathrm{d}v_1 \mathrm{d}v_2 g(v_1) g(v_2) C(v_1 - v_2) \right)^n \\ &\quad = e^{-\frac{1}{2}\int \mathrm{d}v_1 \mathrm{d}v_2 g(v_1) g(v_2) C(v_1 - v_2)}. \end{aligned} \tag{5}$$

Now, I can write $\int \prod_{i=1}^n \delta(\Phi(t_i) - x_i)\mathrm{d}\mu_\mathrm{c}$ in terms of the Fourier transform of the delta function:

$$\begin{aligned} &\int \prod_{i=1}^n \delta(\Phi(t_i) - x_i)\mathrm{d}\mu_\mathrm{c} \\ &= \frac{1}{(2\pi)^n} \int \mathrm{d}\mu_\mathrm{c} \prod_{j=1}^n \mathrm{d}k_j e^{ik_j(\Phi(t_j) - x_j)} \\ &= \frac{1}{(2\pi)^n} \int \prod_{j=1}^n \mathrm{d}k_j e^{-i\sum_{j=1}^n k_j x_j} \int \mathrm{d}\mu_\mathrm{c} e^{i\int \mathrm{d}v \Phi(v) \left[\sum_{j=1}^n k_j \delta(v - t_j)\right]}. \end{aligned} \tag{6}$$

Identifying $g(v) = \sum_{j=1}^n k_j \delta(v - t_j)$ and using Equation **??** gives:

$$\begin{aligned} &\int \prod_{i=1}^n \delta(\Phi(t_i) - x_i)\mathrm{d}\mu_\mathrm{c} \\ &= \frac{1}{(2\pi)^n} \int \prod_{j=1}^n \mathrm{d}k_j e^{-i\sum_{j=1}^n k_j x_j} \\ &\quad \times e^{-\frac{1}{2}\int \mathrm{d}v_1 \mathrm{d}v_2 \sum_{j,\ell=1}^n k_j k_\ell \delta(v_1 - t_j)\delta(v_2 - t_\ell) C(v_1 - v_2)} \\ &= \frac{1}{(2\pi)^n} \int \prod_{j=1}^n \mathrm{d}k_j e^{-i\sum_{j=1}^n k_j x_j} e^{-\frac{1}{2}\sum_{j,\ell=1}^n k_j k_\ell C(t_j - t_\ell)} \\ &= \frac{1}{(2\pi)^n} \int \mathrm{d}\mathrm{k} e^{-i\mathrm{k}^\top \mathrm{x}} e^{-\frac{1}{2}\mathrm{k}^\top \mathrm{C} \mathrm{k}} \end{aligned} \tag{7}$$

where the last line is written in matrix form. Since $\mathrm{C}_{ij} = C(t_i - t_j) = \frac{1}{2m} e^{-m|t_i - t_j|}$ is a real symmetric matrix, it can be diagonalized with an orthogonal matrix $\mathrm{P}^{-1} = \mathrm{P}^\top$ so that $\mathrm{P}\mathrm{C}\mathrm{P}^\top = \Lambda$ and:

$$\begin{aligned} &\int \prod_{i=1}^n \delta(\Phi(t_i) - x_i)\mathrm{d}\mu_\mathrm{c} \\ &= \frac{1}{(2\pi)^n} \int \mathrm{d}\mathrm{k} e^{-i\mathrm{k}^\top \mathrm{P}^\top \mathrm{P} \mathrm{x}} e^{-\frac{1}{2}\mathrm{k}^\top \mathrm{P}^\top \Lambda \mathrm{P} \mathrm{k}} \\ &= \frac{1}{(2\pi)^n} \int \prod_{j=1}^n \mathrm{d}k_j e^{-ik_j (\mathrm{P}\mathrm{x})_j} e^{-\frac{1}{2}k_j{}^2 \Lambda_j} \\ &= \prod_{j=1}^n \frac{1}{\sqrt{2\pi\Lambda_j}} e^{\frac{-(\mathrm{P}\mathrm{x})_j{}^2}{2\Lambda_j}} \\ &= \boxed{\frac{1}{\sqrt{(2\pi)^n \det \mathrm{C}}} e^{-\frac{1}{2}\mathrm{x}^\top \mathrm{C}^{-1} \mathrm{x}}} \end{aligned} \tag{8}$$

When $n = 1$, $C(0) = \frac{1}{2m}$ and $\int \delta(\Phi(t) - x)\mathrm{d}\mu_\mathrm{c} = \sqrt{\frac{m}{\pi}} e^{-mx^2} = |\Omega_0(x)|^2$ the norm-squared of the ground state wave function for the quantum simple harmonic oscillator.