Gaussian Measures and Commutators

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I show how defining a field with a gaussian measure gives rise to a classical interpretation of the equal time commutation relations for a field and its derivative.

One aspect that distinguishes time from space is that while displacements along any spatial axis can be in either the positive or negative direction, our motion in time is monotonic. It is thus possible to consider a space-time that is infinite in spatial extent, but only semi-infinite in temporal extent (i.e. considering functions which are 0 for $t<0$). I will work in Wick-rotated time coordinates for which $\tau = it$ and treat τ as a real coordinate.

I. BACKGROUND FROM A.J.'S 216

Consider the d'Alembertian: $\Box = \partial_t^2 - \Delta$ (this paper will take units in which $\hbar = c = 1$). In terms of τ, this becomes: $\Box = -\partial_\tau^2 - \Delta$. One can study the action of $\Box + m^2$ on $f(\tau, \vec{x})$ by considering its Green's function:

$$C(x-x') = \frac{1}{(2\pi)^4}\int_{\mathbb{R}^4} \frac{e^{ik\cdot(x-x')}}{k^2+m^2}\mathrm{d}E\mathrm{d}\vec{k} \tag{1}$$

where $k = (E, \vec{k})$ is a Euclidian momentum. Defining $\mu(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$, the E integral becomes:

$$\frac{1}{2\pi}\int \frac{e^{iE\tau}}{E^2+\mu^2}\mathrm{d}E = \frac{e^{-|\tau|\mu}}{2\mu} \tag{2}$$

where for this integral to converge, the time t must be treated as imaginary, so that τ is real.

The result is that: $C(x-x') = \left(\frac{1}{2\mu}e^{-|\tau-\tau'|\mu}\right)(\vec{x}-\vec{x}')$ for $\mu = \sqrt{-\Delta + m^2}$. $C(x-x')$ can be used to define a gaussian measure:

$$S(\lambda f) = e^{-\frac{\lambda^2}{2}\langle \bar{f}, Cf\rangle} = \int e^{i\Phi(f)\lambda}\mathrm{d}\mu(\Phi) \tag{3}$$

so that:

$$-\frac{d^2}{d\lambda^2}S(\lambda f)|_{\lambda=0} = \langle \bar{f}, Cf\rangle = \int \Phi(f)^2 \mathrm{d}\mu(\Phi). \tag{4}$$

Requiring that the function f be real and non-zero only for $\tau > 0$ has the advantage of making $\langle \bar{f}, \theta Cf\rangle \geq 0$, where the operator θ inverts τ'.

The above calculations/definitions are based on A.J.'s notes, with the Wick rotation made explicit here. In the following section, I will used the above gaussian measure to show how the canonical commutation relations of quantum fields can be viewed as arising from possible temporal discontinuities in Φ.

II. COMMUTATION RELATIONS FROM THE GAUSSIAN MEASURE

Here, I will integrate out the spatial dependence, and take $\mu \to M$, where M is a constant, mass-like term. The equal time commutation relations of a quantum field are:

$$[\phi(t,\vec{x}), \partial_t\phi(t,\vec{x}')] = i\delta(\vec{x}-\vec{x}');\ [\phi(t,\vec{x}), \phi(t,\vec{x}')] = 0. \tag{5}$$

For a classical field $\Phi(\tau)$ defined by the gaussian measure $\mathrm{d}\mu(\Phi)$, the time correlation is given by $C(\tau-\tau') = \frac{1}{2M}e^{-M|\tau-\tau'|}$, which satisfies $(-\partial_\tau^2 + M^2)C(\tau-\tau') = \delta(\tau-\tau')$ by its definition as a Green's function.

In the context of a random, not necessarily continuous, field $\Phi(\tau)$, the idea of a local time derivative should be replaced with the limit definition:

$$\partial_t\Phi(\tau) = i\partial_\tau\Phi = i\lim_{\Delta\tau\to 0}\frac{\Phi(\tau+\Delta\tau)-\Phi(\tau)}{\Delta\tau} \tag{6}$$

where this limit takes physical meaning when its expectation value with respect to $\mathrm{d}\mu(\Phi)$ is taken.

If one interprets $\langle\Omega|[\phi, \partial_t\phi]|\Omega\rangle$ as the difference of 1: measuring the time derivative and then the field, and 2: measuring the field and then its time derivative, in the limit at which the field and derivative measurements approach being at the same time, then the necessity of specifying such a time order becomes natural in the context of a derivative that is defined in terms of a limit of two field measurements spaced by $\Delta\tau$. For $\Phi(\tau)$:

$$\begin{aligned}
&\int \Phi(\tau^+)\partial_t\Phi(\tau^-) - \partial_t\Phi(\tau^+)\Phi(\tau^-)\mathrm{d}\mu(\Phi) \\
&= i \lim_{\Delta\tau\to 0}\int \Phi(\tau+\Delta\tau)\tfrac{\Phi(\tau+\Delta\tau)-\Phi(\tau)}{\Delta\tau} \\
&\qquad - \tfrac{\Phi(\tau+\Delta\tau)-\Phi(\tau)}{\Delta\tau}\Phi(\tau)\mathrm{d}\mu(\Phi) \\
&= i \lim_{\Delta\tau\to 0}\int \tfrac{(\Phi(\tau+\Delta\tau)-\Phi(\tau))^2}{\Delta\tau}\mathrm{d}\mu(\Phi) \\
&= 2i \lim_{\Delta\tau\to 0}\tfrac{[C(0)-C(\Delta\tau)]}{\Delta\tau} \\
&= \tfrac{i}{M}\lim_{\Delta\tau\to 0}\tfrac{[1-e^{-M\Delta\tau}]}{\Delta\tau} \\
&= i
\end{aligned} \tag{7}$$

Since $\delta(\tau-\tau') = -i\delta(t-t')$, $C(x-x')$ satisfies the same differential equation as the Feynman propagator:

$$(\Box + m^2)\int \Phi(x)\Phi(x')\mathrm{d}\mu\Phi = -i\delta^4(x-x'). \tag{8}$$

A classical expectation value which behaves like the Feynman propagator gives a classical interpretation of $[\phi, \partial_t\phi]$. The ordering of $\Phi(x)$ and $\partial_t\Phi(x)$ matters because of the non-locality of measuring a time averaged derivative of a function that is Hölder continuous with exponent 1/2.