Covariant Derivatives and the Hamilton-Jacobi Equation

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I define a covariant derivative to simplify how higher order derivatives act on a classical generating function.

When studying the connection between classical and quantum mechanics, it would be nice to have a differential operator which, when acting repeatedly on some function $e^{i\frac{S(q,P,t)}{\hbar}}$ pulls down powers of the derivatives of the function within the exponent.

Consider the results of "Wavefunctions and the Hamilton-Jacobi Equation." There, I performed a canonical change of variables from (q_i, p_i) to constants (Q_i, P_i):

$$p_i\dot{q}_i - H(q,p,t) = P_i\dot{Q}_i - K(Q,P,t) + \frac{dF}{dt} \tag{1}$$

where $F = S(q,P,t) - P_iQ_i$, and found:

$$H = -\frac{\partial S}{\partial t} \quad p_i = \frac{\partial S}{\partial q_i} \quad Q_i = \frac{\partial S}{\partial P_i}. \tag{2}$$

when $K = 0$. At first order, the function $e^{i\frac{S}{\hbar}}$ had the property that ordinary multiplication by the value of H or p_j was equivalent to acting with a differential operator:

$$\begin{aligned} H \cdot e^{i\frac{S}{\hbar}} &= -\tfrac{\partial S}{\partial t}e^{i\frac{S}{\hbar}} = i\hbar\partial_t e^{i\frac{S}{\hbar}} \\ p_j \cdot e^{i\frac{S}{\hbar}} &= \tfrac{\partial S}{\partial q_j}e^{i\frac{S}{\hbar}} = \tfrac{\hbar}{i}\partial_{q_j}e^{i\frac{S}{\hbar}}. \end{aligned} \tag{3}$$

Moreover, this connection between multiplication and a differential operator held at first order for arbitrary superpositions:

$$\Phi(q,t) = \int \mathbb{P}(P_i)e^{i\frac{S(q,P,t)}{\hbar}}\mathrm{d}P_i. \tag{4}$$

In this paper, I will consider the one dimensional case $q = x$ and define a covariant derivative such that acting on $e^{i\frac{S(x,P,t)}{\hbar}}$ a total of n times with the operator $\frac{\hbar}{i}\nabla_x$ is exactly equivalent to multiplying $e^{i\frac{S(x,P,t)}{\hbar}}$ by $p^n = (\frac{\partial S}{\partial x})^n$.

The key is to treat $e^{i\frac{S(x,P,t)}{\hbar}}$ as a scalar, with a non-trivial one-dimensional spatial metric $g_{xx} = (\frac{\partial S}{\partial q})^2$. Then there is a non-zero connection $\Gamma^x_{xx} = \frac{1}{2}g^{xx}\partial_x g_{xx} = \frac{S''}{S'}$, where primes denote partial derivatives with respect to x.

If I treat $\hat{p}^n e^{i\frac{S(x,P,t)}{\hbar}} \equiv \left(\frac{\hbar}{i}\right)^n \nabla_n e^{i\frac{S(x,P,t)}{\hbar}}$ as a covariant rank-n tensor, I find that:

$$\begin{aligned} \nabla_n e^{i\frac{S}{\hbar}} &\equiv \nabla_x\nabla_x ... \nabla_x e^{i\frac{S}{\hbar}} \\ &= \partial_x(\nabla_{n-1}e^{i\frac{S}{\hbar}}) - (n-1)\Gamma^x_{xx}\nabla_{n-1}e^{i\frac{S}{\hbar}} \end{aligned} \tag{5}$$

It is quick to check for $n = 1$ that $\nabla_1 e^{i\frac{S}{\hbar}} \equiv \nabla_x e^{i\frac{S}{\hbar}} = \partial_x e^{i\frac{S}{\hbar}}$. If it is true that $\nabla_{n-1}e^{i\frac{S}{\hbar}} = (\frac{i}{\hbar}S')^{(n-1)}e^{i\frac{S}{\hbar}}$, then:

$$\begin{aligned} \nabla_n e^{i\frac{S}{\hbar}} &= \partial_x((\tfrac{i}{\hbar}S')^{(n-1)}e^{i\frac{S}{\hbar}}) - (n-1)\Gamma^x_{xx}\cdot(\tfrac{i}{\hbar}S')^{(n-1)}e^{i\frac{S}{\hbar}} \\ &= (n-1)(\tfrac{i}{\hbar}S')^{(n-2)}\tfrac{i}{\hbar}S''e^{i\frac{S}{\hbar}} + (\tfrac{i}{\hbar}S')^n e^{i\frac{S}{\hbar}} \\ &\quad - (n-1)\Gamma^x_{xx}\cdot(\tfrac{i}{\hbar}S')^{(n-1)}e^{i\frac{S}{\hbar}} \\ &= (\tfrac{i}{\hbar}S')^n e^{i\frac{S}{\hbar}} \end{aligned} \tag{6}$$

since $\Gamma^x_{xx} = \frac{S''}{S'}$, so $\left(\frac{\hbar}{i}S'\right)^n e^{i\frac{S(x,P,t)}{\hbar}} = \hat{p}^n e^{i\frac{S(x,P,t)}{\hbar}} = \left(\frac{\hbar}{i}\right)^n \nabla_n e^{i\frac{S(x,P,t)}{\hbar}}$ holds by induction.

If I define a partition function expectation value:

$$\langle\mathcal{O}(x,p)\rangle \equiv \frac{\int \mathbb{P}(P)\mathrm{d}P\mathrm{d}q\mathcal{O}(x,p)e^{i\frac{S(x,P,t)}{\hbar}}}{\int \mathbb{P}(P)\mathrm{d}P\mathrm{d}qe^{i\frac{S(x,P,t)}{\hbar}}} \tag{7}$$

then this is equivalent to:

$$\langle\mathcal{O}(x,p)\rangle = \frac{\int \mathbb{P}(P)\mathrm{d}P\mathrm{d}x : \hat{\mathcal{O}}(x,\hat{p}) : e^{i\frac{S(x,P,t)}{\hbar}}}{\int \mathbb{P}(P)\mathrm{d}P\mathrm{d}xe^{i\frac{S(x,P,t)}{\hbar}}} \tag{8}$$

where the normal ordered operator is defined such that all of the momentum operators appear on the right. The direct correspondence between $x^n p^m = x^n(S')^m$ in $\mathcal{O}$ and $x^n\hat{p}^m = x^n(\frac{\hbar}{i})^m\nabla_m$ in $:\hat{\mathcal{O}}(q,\hat{p}):$ thus follows from the composition property of the covariant derivative.

Summarizing Equation 8 as $\langle\mathcal{O}(q,p)\rangle = \langle:\hat{\mathcal{O}}(q,\hat{p}):\rangle$, one finds that for an operator which does not explicitly depend on time:

$$\begin{aligned} \langle\tfrac{\mathrm{d}\mathcal{O}}{\mathrm{d}t}\rangle &= \langle\{\mathcal{O},H\}\rangle \\ &= \langle:\widehat{\{\mathcal{O},H\}}:\rangle \\ &= \tfrac{-i}{\hbar}\langle:[:\hat{\mathcal{O}}:,:\hat{H}:]:\rangle \end{aligned} \tag{9}$$

The last equality comes from considering a generic term in the series expansion of $\mathcal{O}(x,p)$

$$\begin{aligned} \langle\{x^np^m, x^rp^s\}\rangle &= (ns-mr)\langle x^{n+r-1}p^{m+s-1}\rangle \\ &= (ns-mr)\langle x^{n+r-1}\hat{p}^{m+s-1}\rangle \end{aligned} \tag{10}$$

versus

$$\begin{aligned} \tfrac{-i}{\hbar}\langle:[x^n\hat{p}^m, x^r\hat{p}^s]:\rangle &= \langle:x^n[\hat{p}^m,x^r]\hat{p}^s + x^r[x^n,\hat{p}^s]\hat{p}^m:\rangle \\ &= (ns-mr)\langle x^{n+r-1}\hat{p}^{m+s-1}\rangle \end{aligned} \tag{11}$$

where some care must be taken when specifying what it means to normal order the commutator (e.x. I would want to have $:[x,\hat{p}]:= i\hbar$ and not $:[x,\hat{p}]:=: x\hat{p}: - :\hat{p}x := 0$).

Equation 9 is similar to Ehrenfest's Theorem. There is a natural association between the Poisson Bracket of classical mechanics and the normal ordered commutator of normal ordered operators.