Space and Time

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I arrive at Maxwell's Equations, gauge invariance, and the force law for charges, starting from a conserved current.

Electricity and magnetism can be motivated from two equations:

$$\partial_\mu J^\mu = 0 \quad \& \quad v_\mu F^\mu = 0. \tag{1}$$

The first, $\partial_\mu J^\mu = 0$, states that there exists a conserved current and is a compact expression for the continuity equation:

$$J^\mu = \begin{pmatrix} \rho c \\ \rho v_x \\ \rho v_y \\ \rho v_z \end{pmatrix} \quad \rightarrow \quad \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{v} = 0. \tag{2}$$

The second, $v_\mu F^\mu = 0$, is consistent with the definition of the four-force, F^μ, as the derivate with respect to proper time τ (see "Conformal Mapping of Displacement Vectors") of the energy-momentum four-vector. Plugging in $p^\mu = mv^\mu$:

$$v_\mu F^\mu = \frac{1}{m} p_\mu \frac{\mathrm{d}}{\mathrm{d}\tau} p^\mu = \frac{1}{2m} \frac{\mathrm{d}}{\mathrm{d}\tau} p_\mu p^\mu \quad \propto \quad \frac{\mathrm{dm}}{\mathrm{d}\tau} = 0. \tag{3}$$

The defining feature of these equations is that they are valid in any inertial reference frame. While the four-vectors within these expressions transform during boosts (as discussed in "Motivating Special Relativity using Linear Algebra"), the contraction of indices gives a Lorentz invariant (see "Dot Products in Special Relativity").

I focus on an approach that employs determinants rather than Levi-Civita tensor notation and extends the concept of finding orthogonal vectors described in "An Alternate Approach for Finding an Orthogonal Basis." An important starting point is that the divergence of a curl is zero. Looking at the expression:

$$\vec{\nabla} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ B_x & B_y & B_z \end{vmatrix} \tag{4}$$

and noting that the dot product with a constant vector: $\vec{C} \cdot (\vec{\nabla} \times \vec{B})$, is equivalent to replacing the unit vectors in the first row with $\vec{C}$, gives a helpful mnemonic for remembering that the divergence of the cross product is zero. However, the reason why this is true is not because $\vec{\nabla}$ can be thought of as a vector that is parallel to itself, but rather because partial derivatives commute and the determinant produces antisymmetric combinations of the entries.

Electricity and magnetism arises from finding an expression for J^μ. The core idea is that J^μ can be expressed in terms of a single four-potential A^μ. In the following calculations, J^μ will be constructed from A^μ and other objects that are independent of the system: ∂^μ and the Minkowski metric.

The constraint is that $\partial_\mu J^\mu = 0$ from Equation 1. Because there are four components, it is not possible to use an ordinary curl, as in Equation 4. The idea from my orthogonal vector paper of using a 4×4 determinant can be applied, however. The one extension needed is that the unit vector for the time direction picks up a negative sign relative to the spatial unit vectors to be consistent with the modified inner product. One vector that has zero divergence is:

$$K^\mu(\nu) = \begin{vmatrix} -\hat{t} & \hat{x} & \hat{y} & \hat{z} \\ \nu^0 & \nu^1 & \nu^2 & \nu^3 \\ \partial^0 & \partial^1 & \partial^2 & \partial^3 \\ A^0 & A^1 & A^2 & A^3 \end{vmatrix} \tag{5}$$

for any constant four-vector ν^μ (so that the derivatives only hit A^μ). This is not sufficient to define J^μ, however, because there should not be a special direction ν^μ. Instead, I use K^μ to find another vector that is divergenceless. Wanting J^μ to be linear in A^μ, I replace the A^μ row with K^μ:

$$H^\mu(\nu, \chi) = \begin{vmatrix} -\hat{t} & \hat{x} & \hat{y} & \hat{z} \\ \chi^0 & \chi^1 & \chi^2 & \chi^3 \\ \partial^0 & \partial^1 & \partial^2 & \partial^3 \\ K^0 & K^1 & K^2 & K^3 \end{vmatrix} \tag{6}$$

to get a vector that involves products of the components of ν^μ and χ^μ. This suggests a generalization of Equations 5 and 6. Rather than including two arbitrary vectors, ν^μ and χ^μ, I could have their components represent unit vectors. Products of different components would then correspond to inner products of these unit vectors and the invariance of the Minkowski metric would allow an expression for J^μ in terms of only derivatives of A^μ. In place of Equation 5, I get the tensor:

$$G^{\mu\nu} = \begin{vmatrix} -\hat{t}_\mu & \hat{x}_\mu & \hat{y}_\mu & \hat{z}_\mu \\ -\hat{t}_\nu & \hat{x}_\nu & \hat{y}_\nu & \hat{z}_\nu \\ \partial^0 & \partial^1 & \partial^2 & \partial^3 \\ A^0 & A^1 & A^2 & A^3 \end{vmatrix}. \tag{7}$$

Substituting the coefficients of the μ unit vectors in place of the A^μ row, as in Equation 6, and contracting over the ν index by taking inner products of ν unit vectors that are multiplied together gives:

$$J^{\mu} \propto -\tfrac{1}{2}\begin{vmatrix} -\hat{t}_{\mu} & \hat{x}_{\mu} & \hat{y}_{\mu} & \hat{z}_{\mu} \\ -\hat{t}_{\nu} & \hat{x}_{\nu} & \hat{y}_{\nu} & \hat{z}_{\nu} \\ \partial^0 & \partial^1 & \partial^2 & \partial^3 \\ G^{0\nu} & G^{1\nu} & G^{2\nu} & G^{3\nu} \end{vmatrix} \tag{8}$$

$$= \begin{pmatrix} -\nabla^2 A^0 - \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) \\ \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}A^0 + \frac{1}{c}\frac{\partial}{\partial t}\vec{A}) \end{pmatrix}$$

Where the overall scaling does not affect the divergence-less property of J^{μ}.

Defining $\vec{E} = -\vec{\nabla}A^0 c - \partial_t \vec{A}$ and $\vec{B} = \vec{\nabla}\times\vec{A}$ as the electric and magnetic fields, with $A^0 = \phi/c$ as the static electric potential, Equation 8 gives the inhomogeneous Maxwell equations:

$$\begin{gathered} \vec{\nabla}\cdot\vec{E} = \tfrac{\rho}{\epsilon_0} \\ \vec{\nabla}\times\vec{B} - \tfrac{1}{c^2}\tfrac{\partial}{\partial t}\vec{E} = \mu_0\vec{J}. \end{gathered} \tag{9}$$

The divergence-less property of $G^{\mu\nu}$ gives the homogeneous Maxwell Equations:

$$\begin{gathered} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} + \tfrac{\partial}{\partial t}\vec{B} = \vec{0}. \end{gathered} \tag{10}$$

The gauge invariance of the vector potential is evident from the way in which J^{μ} was derived using determinants. Determinants are linear in each row in the sense that if one row includes the vector $\vec{v} = \vec{u}_1 + \vec{u}_2$, the total determinant is equal to the sum of the determinants found by replacing $\vec{v}$ with $\vec{u}_1$ and then $\vec{v}$ with $\vec{u}_2$. The result of Equation 7 would be the same for $A'^{\mu} = A^{\mu} + \lambda^{\mu}$, if $\lambda^{\mu} = \partial^{\mu} f$ for some function $f(ct, x)$, since the determinant with $\partial^{\mu} f$ as the last row is zero.

The step that was used to get to Equation 8 from Equation 7 can also be used to arrive at the force law for charged particles. Equation 1 says that $v_{\mu}F^{\mu} = 0$. Rather than needing to find a vector whose inner product with ∂^{μ} is zero, the goal now is to get a vector whose inner product with v^{μ} is zero:

$$F^{\mu} \propto -\tfrac{1}{2}\begin{vmatrix} -\hat{t}_{\mu} & \hat{x}_{\mu} & \hat{y}_{\mu} & \hat{z}_{\mu} \\ -\hat{t}_{\nu} & \hat{x}_{\nu} & \hat{y}_{\nu} & \hat{z}_{\nu} \\ v^0 & v^1 & v^2 & v^3 \\ G^{0\nu} & G^{1\nu} & G^{2\nu} & G^{3\nu} \end{vmatrix} \tag{11}$$

$$= \gamma \begin{pmatrix} \frac{1}{c}\vec{v}\cdot\vec{E} \\ \vec{E} + \vec{v}\times\vec{B} \end{pmatrix}$$

where the same contraction of ν unit vectors is performed as was described earlier for finding J^{μ}. With the charge q as the proportionality constant, this returns the Lorentz force law:

$$\vec{F} = q\left[\vec{E} + \vec{v}\times B\right]. \tag{12}$$

In summary, I arrived at results from electricity and magnetism by requiring that: 1) a divergence-less current be described by second derivatives of a vector potential; and 2) that the four-force for a charge be proportional to first derivatives of this potential and orthogonal to the four-velocity of that charge. In this method, the difference between $\vec{E}$ and $\vec{B}$ appears as a difference between space and time. While $\vec{B} = \vec{\nabla}\times\vec{A}$ takes crossed spatial derivatives of the spatial components of $\vec{A}$, I found that $\vec{E}$ combines spatial derivatives of the time-like component with the time derivatives of the corresponding spatial components. This highlights a symmetry between interchanging space and time, rather than $\vec{E}$ and $\vec{B}$.

APPENDIX

I will now use Einstein summation notation to reach the same results from a different route. The above derivation illustrates a structure to the way in which the fields and four-potential are defined relative to one another: each field mixes only two of the four space-time coordinates of the four-potential, so that there are $\binom{4}{2} = 6$ field components. This can be imbedded in the way that the potential is defined if I postulate that currents and forces can arise from taking derivatives or multiplying four-velocities by a tensor potential:

$$\mathbb{A}^{\mu\nu}_{\sigma} = \frac{1}{2}\varepsilon^{\mu\nu\gamma\alpha}\varepsilon_{\gamma\alpha\sigma\tau}A^{\tau} \tag{13}$$

which restricts the set $\{\sigma, \tau\}$ to the set $\{\mu, \nu\}$. The anti-symmetry of $\mu \leftrightarrow \nu$ means there are six possible ways to form distinct scalars by contracting with combinations of ∂^{μ} and v^{μ}:

$$\begin{gathered} \partial_{\mu}\partial_{\nu}\partial^{\sigma}\mathbb{A}^{\mu\nu}_{\sigma} = 0,\ \ v_{\mu}v_{\nu}\partial^{\sigma}\mathbb{A}^{\mu\nu}_{\sigma} = 0,\ \ v_{\mu}\partial_{\nu}\partial^{\sigma}\mathbb{A}^{\mu\nu}_{\sigma} \neq 0; \\ \partial_{\mu}\partial_{\nu}v^{\sigma}\mathbb{A}^{\mu\nu}_{\sigma} = 0,\ \ v_{\mu}v_{\nu}v^{\sigma}\mathbb{A}^{\mu\nu}_{\sigma} = 0,\ \ v_{\mu}\partial_{\nu}v^{\sigma}\mathbb{A}^{\mu\nu}_{\sigma} \neq 0. \end{gathered} \tag{14}$$

Since $\partial_{\mu}J^{\mu} = 0$, the first column in Equation 14 gives $\partial_{\nu}\partial^{\sigma}\mathbb{A}^{\mu\nu}_{\sigma}$ and $\partial_{\nu}v^{\sigma}\mathbb{A}^{\mu\nu}_{\sigma}$ as possible candidates for J^{μ}. The second option is excluded by requiring that the four-current be independent of the four-velocity of an external charge carrier.

Since $v_{\mu}F^{\mu} = 0$, the second column in Equation 14 gives $v_{\nu}\partial^{\sigma}\mathbb{A}^{\mu\nu}_{\sigma}$ and $v_{\nu}v^{\sigma}\mathbb{A}^{\mu\nu}_{\sigma}$ as possible candidates for F^{μ}. Requiring that the force involve derivatives of the potential eliminates the second option. In terms of $\mathbb{A}^{\mu\nu}_{\sigma}$, I then have:

$$\partial^{\sigma}\mathbb{A}^{\mu\nu}_{\sigma} = -\mathcal{F}^{\mu\nu},\ \ \partial_{\nu}\partial^{\sigma}\mathbb{A}^{\mu\nu}_{\sigma} = \mu_0 J^{\mu},\ \ v_{\nu}\partial^{\sigma}\mathbb{A}^{\mu\nu}_{\sigma} = \frac{1}{q}F^{\mu} \tag{15}$$

the Faraday Tensor, four-current, and force law.